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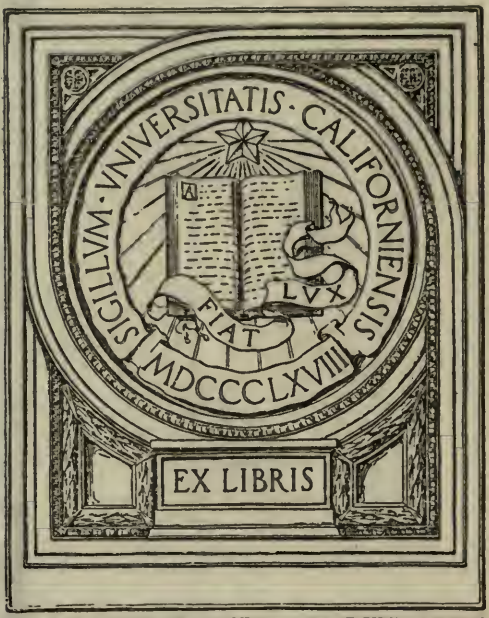

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METHODS

IN

Written Arithmetic,

BY

7227

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ILLINOIS STATE NORMAL UNIVERSITY.

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CONTENTS.

PAGE.

Preface, ... 5

Definitions, Notation, Numeration, ... 9

Addition, ... 19

Subtraction, ... 25

Multiplication, ... 29

Division, ... 37

Factoring, ... 46

Cancellation, ... 56

Multiples, ... 64

Fractions, ... 68

Addition of Fractions, ... 74

Division and Partition of Fractions, ... 80

The Method of Finding the Part which one Number is of Another, ... 87

Decimal Fractions, ... 93

Contents.

	PAGE.
Compound Numbers,	100
Measures of Length,	109
Longitude of Time,	118
Percentage,	126
Loss and Gain,	134
Commission,	139
Stock Investments,	145
Interest,	150
Partial Payments,	158
Present Worth,	160
Bank Discount,	163
Exchange,	170
Equation of Payments,	179
Evolution	182

PREFACE.

It has long been the conviction of the writer of the following pages that the subject of arithmetic, as it is too often studied, yields a smaller return for the time devoted to it, than any other branch included in the curriculum of our common schools. Pupils begin the study of numbers at the threshold of their school life and continue it for from six to ten years; yet a large proportion of them have but little appreciation of the principles of the science, and their work is, consequently, almost entirely mechanical.

A subject that receives so large a share of the pupils' time should accomplish something more in the direction of real education than to give the power to make simple calculations with tolerable accuracy.

If the subject is left without having given the child the ability to define with sharp discrimination,

to analyze with fluency, and to give the reasons for the operations involved, the work must be regarded as, at best, but a partial success.

This little volume does not pretend to be an exhaustive treatise. Its purpose is simply to present detailed methods of work in the more familiar parts of the subject, to suggest forms of analysis that seem to be reasonably acceptable, to state as clearly as possible the leading principles involved, and to urge the necessity of *thought work* in the arithmetic class. No attempt is made to furnish drill problems. It is expected that the book will be used in connection with some regular text-book as a supplementary work in analysis, etc. It owes its appearance to the requests of many normal students who desired to see certain features of their work in arithmetic put into permanent and accessible form.

With the hope that it may be helpful to some, at least, of my fellow teachers, it is herewith submitted.

NORMAL, ILLINOIS.

7227

METHODS

IN

WRITTEN ARITHMETIC.

CHAPTER I.

ARITHMETIC is the science which treats of number, its nature and properties, the methods of writing, reading, combining and resolving it, and of its application to practical affairs.—*Olney.*

Number is that which answers the question, "How many?"

Number is one, or is formed by the repetition of ones.

Ones are either Absolute or Relative.

The Absolute unit is without limitation.

Relative units are units formed by uniting Absolute units, or by the separation of an Absolute unit into equal parts.

Numbers are either Integral or Fractional.

An Integral uumber is a number that is composed of one or more Integral units.

An Integral unit is an Absolute unit or a Relative unit formed by uniting Absolute units.

A Fractional number is a number that is composed of one or more Fractional units.

A Fractional unit is a Relative unit that is formed by the separation of an Absolute unit into equal parts.

But three operations are possible with numbers. They may be expressed, united, and separated.

Numbers may be expressed by characters, or orally.

Notation treats of the methods of expressing numbers by means of characters.

Numeration treats of the methods of reading num bers that are expressed by figures.

There are three methods of Notation: (1.) The Word Method; (2.) The Letter Method; (3.) The Figure Method.

The Word Method is used in general correspondence, essays, and similar productions.

The Letter Method is usually employed in paging prefaces and appendixes; in numbering volumes, lessons, and chapters; in the titles of monarchs; on the dials of time-pieces, etc.

It employs the printed letters I, V, X, L, C, D, M, and the dash.

I=1, V=5, X=10, L=50, C=100, D=500, M=1000. Both the capitals and lower-case letters are used. The scale is five and two; thus five I's=one V; two V's=one X, etc.

LAWS.

1. Repeating a letter repeats its value.

2. When a letter is placed before one of greater value, the two express a number equal to the difference of their values.

3. When a letter is placed after one of greater value, the two express a number equal to the sum of their values.

4. When a letter is placed between two, each of greater value, the three express a number equal to the difference of the values of the less and of the sum of the two greater.

5. Placing a dash over a letter multiplies its value by one thousand.

Pupils of the grade implied by this work should learn to read and write readily at least as far as M M, and usually much farther.

The Figure, or Arabic Method of notation employs ten characters. They are: 1, 2, 3, 4, 5, 6, 7, 8, 9, 0.

These characters are called digits, from the Latin *digitus*, meaning finger.

The first nine are called significant figures. They have two values: an absolute or shape value, and a relative or place value.

The zero is used to fill vacant places between some significant figure and the decimal point.

The decimal point is used to fix the order of absolute units. In doing so it also determines the names of relative units.

A scale is the statement of the number of units in each order required to make one of the next higher.

A decimal system of numbers is one in which the scale is ten.

There are two systems of Numeration: The French and the English.

The former is the one in common use in this country. It groups figures into periods of three orders each, beginning at the right. It enables the reader to tell *how many* are expressed by the succession of figures.

The right-hand place, or order, in each group is called units' order; the second, tens'; the third, hundreds'.

Each group is read as if standing alone; then the name of the period is spoken, and, at the close, the kind of units numbered.

Notation and Numeration are taught together.

Pupils should be able to read and write with perfect readiness all numbers from quadrillions to billionths. When readiness has been acquired in this work, pupils can extend it at liberty.

Teachers have noticed the trouble that pupils experience, in writing, in determining whether their work is correct. They stand with crayon in one hand and eraser in the other, and alternately write, "numerate" and erase. The writing should be done rapidly, no erasing should be permitted, and the instant the writing is completed the pupil should turn from the board. In order to do this, certain knowledge is necessary.

1. The pupil must know the order of periods, from *left* to *right*.

2. He should know the number of each period at the left of the point, and of each order at the right.

The number of the period can be fixed easily by observing the meaning of the prefixes in the names.

The force of the *bi* in billions is *two;* of the *tri* in trillions is *three*, and so on.

Observe that:

The number of any period above millions is two more than the meaning of the prefix in the name of the period.

The numbers of the orders at the right of the point should be thoroughly memorized, so that the pupil can speak the number if the name is given, or *vice versa*.

3. The instant that a period is filled it should be followed by a comma, unless it is units' period.

Suppose a pupil is asked to write 17 trillions, 286 millions, 521. He writes 17 and follows it with a comma. He remembers that trillions' period is the fifth, and millions' is the third, so he writes three zeroes and a comma, then 286 and a comma. Remembering that thousands' period is the second, he

writes three zeroes and a comma, and then 521 and the period. In the early work in writing it is wise to insist upon the invariable use of the period.

If the pupil is asked to write 624 millionths, he should remember that the 4 must stand in the sixth order at the right; hence he writes the period, three zeroes and 624. He has no need to "numerate," and it should not be permitted.

In reading numbers the word *and* should not be used except with expressions that lie on both sides of the point, and then only when read as mixed numbers. Thus, 56.693 should be read as fifty-six *and* six hundred ninety-three thousandths (a mixed number), or as fifty-six thousand six hundred ninety-three thousandths (a simple number).

EXERCISES.

1. To ascertain whether the pupils have memorized the numbers of the periods and orders, Dictation Exercises, like the following, may be given: With the class at the board, or supplied with slates, the teacher rapidly names the periods, thus: millions, sextillions, thousands, trillions, etc., and the pupils write, 3, 8, 2, 5, etc.

2. Give a similar exercise for orders at the right of the point.

3. In writing large numbers, let the teacher read a single period and wait for the pupil to write, then another, etc.

4. Vary exercise (3) by requiring the class to face the teacher while he speaks the number *once*, then at a signal class turns and writes, putting in commas, doing no erasing nor "numerating," and "facing" immediately when the work is completed.

5. Instead of exercise (2) give the following occasionally: Give the number to be written without naming the kind and without permitting the voice to fall. When it is written speak the denomination and the pupils put the period into its proper position.

6. Have the pupils "write without the crayon," thus: They read "63 trillions, 589 millions, 274," and then say: 63, comma, three zeroes, comma, 589 comma, three zeroes, comma, 274, period." Use the same exercise in writing decimals.

7. In preparing numbers for reading, the names of the periods should be spoken as the commas are placed, and the reading should be begun as soon as the "pointing" is completed.

It is of the utmost importance that pupils acquire the habit of picturing the exact appearance of an expression before making any efforts at writing. Thus, in writing decimal fractions they should think, 1. Of the number of figures necessary to express the *how many* there are in the number; 2. Where the right-hand figure must stand in order that the denomination may be expressed properly; 3. How many zeroes must be placed between the decimal point and the first significant figure to make the right-hand figure stand in the proper order.

If, for example, they are told to write 5608 ten-millionths, they should notice, (1) that four figures are necessary to express 5608; (2) that the 8 must stand in the seventh order at the right of the point to make the number ten-millionths, and (3) that three zeroes must stand between the point and the 5; hence they write, decimal point, three zeroes, 5608.

CAUTIONS.

1. Names of orders are often given incorrectly. Say "ten-thousandths," not "tens of thousandths;" "hundred-millionths," not hundreds of millionths."

2. When the word *period* or *order* is understood

after the noun, the noun should be written with the apostrophe: billionths' period; thousandths' order.

3. See that all board work is scrupulously neat.

4. Allow no one to use an eraser without permission.

The English System of Numeration has passed out of use in this country, but the names are retained in the French System, although their meaning has been lost.

The English System employs six orders in a period. The name of the first period is units; the second, millions; the third, billions, etc.

Million is derived from *mille*, a thousand. It is a thousand thousands.

Billion is contracted from *bis*, twice, and *million*. It is the square of a million.

Trillion is from *tres*, three, and *million*. It is the third power of a million; and so following.

In the French System these terms, except *million*, are used arbitrarily, a billion being a thousand millions, and a trillion a thousand billions.

CHAPTER II.

NUMBERS are united in two ways: (1.) By Addition; (2.) By Multiplication. The second may be regarded as a special case in Addition.

The grade of the work discussed herein should not be forgotten. The pupils are supposed to have done an amount of work represented by the First Book in Number. It should be remembered that proficiency of the pupils will depend upon the amount of work that each does for himself, hence it should be the aim of the teacher to keep every member of the class as busy as possible during the recitation.

Try a teachers' Institute on a problem in Addition which involves a dozen three-place numbers, and ten per cent. of the results will be incorrect. The reason is obvious—they have not the technical knowledge necessary for rapid and correct work. The memory must be stored with facts, and in this case each of the facts is the sum of two numbers.

The ground facts to be thoroughly mastered are the sums of any two numbers each of which is less than ten. These are contained in the Addition tables of any First Book. When a pupil knows that the sum of 8 and 9 is 17, he can readily learn that the sum of 9 and any two-place number ending in 8 has 7 for its right-hand digit, and that the number of tens is increased by one.

Addition is the process of finding the sum of two or more like numbers.

The sum of two or more numbers is a number which contains as many units as all of the numbers taken together.

EXERCISES.

Each pupil should be provided with a slate and pencil, or their equivalent.

1. The teacher stands before the class and pronounces eight or ten one-place numbers as rapidly as the class can work "mentally." When his voice falls, the pupils write the result. The same exercise is repeated until six or eight problems have been performed, the teacher meanwhile keeping a list of the several sums. The results should then be examined. This exercise need not occupy more than three min-

utes. Pupils should not be permitted to see each other's work.

2. Send the class to the board. Give several dictation exercises in two-place, or larger numbers, the pupils writing all the numbers.

In order that they may get no aid from each other, let them number in order and assign one problem to the odd numbers and another to the even.

Permit no erasing and no changing. Hold all responsible for the first result. Insist that the work shall be scrupulously neat. Carry the work to the right of the point as in the writing suggested in the previous chapter.

3. Write upon the board a line of figures and one figure below the line, thus:

3 8 9 6 7 4 5 6 9 3

8

Require one pupil to give the sum of 8 and 3; another of 8 and 8, and so on around the class. This should go very rapidly.

The figures may be written as follows, and another be placed within, thus:

By this means the exercise can go on indefinitely without interruption. Pupils should not be permitted to name the numbers added, but should give the results only.

4. Introduce competitive exercises. These must be used with care, as some pupils are easily confused or discouraged. Multiply exercises until the pupils are stimulated to do considerable practice-work out of the class.

LANGUAGE WORK.

Results in arithmetic are of no account unless correct. Accuracy is the first essential. Equal definiteness should be secured in the language work. He misses the best culture that the study of arithmetic should afford who leaves it without the power to talk clearly and pointedly.

Definitions should be memorized with extreme care. Analyses should be given promptly and accurately, and without the aid of suggestive questions. The pupils should be required to go through a complete explanation while the teacher simply listens. Much patient labor is required to reach such a result, but no teacher should be satisfied with less. Pupils may be aided at first by judicious questions, but help should be withdrawn as soon as possible, and they should be required to work alone.

Give them a chance to do something for themselves.

Faulty expressions, too many of which are found in some text-books, should be corrected. Figures are not numbers any more than a photograph is a man, hence they cannot be added. Pupils talk about writing figures "under each other." How many writings of each would be required to place 2 and 5 under each other? "The sum of 9 and 7 *would be* 16," is often heard. What is implied in such an expression?

A statement somewhat like the following should be secured:

For convenience I write the numbers with units of the same order in the same column. Beginning with the lowest order I find the sum of the numbers

expressed in each column. If the sum in any case is nine or less, I express it under the column added; if more than nine, I reduce it to the next higher order, writing the remainder, if any, under the column added, and adding the reduced number to the first number expressed in the next column, etc.

QUESTIONS.

Is it necessary to write the numbers as above? Why more convenient? Is it necessary to begin with the lowest order? When is it equally convenient to begin with the highest? When is it more convenient to begin with the lowest? Why is it more convenient?

The so-called "proofs" are only checks against errors. Addition in the opposite direction is the simplest test of accuracy.

An ordinary problem in addition includes a number of problems in each of the fundamental processes. See that the pupils recognize this fact,

Repetition is the secret of success. He adds well who has added a great deal.

CHAPTER III.

THERE are two methods of resolving numbers: 1. Subtraction, and 2. Division.

Subtraction is the process of separating a number into two parts, one of which is given for the purpose of finding the other part.

The definition may be verified by performing a problem with objects.

"If John had eight sticks and gave three of them to Thomas how many had he left?"

The problem frequently assumes a different form:

"If John had eight sticks and Thomas three, John had how many more than Thomas?"

The solution obviously is the same as in the former problem. Separate John's sticks into two parts, one of which shall contain as many sticks as Thomas had. The remainder is found in the same manner as before.

Where pupils have acquired a knowledge of subtraction by performing the operations with objects, there will be no confusion of ideas.

DEFINITIONS OF TERMS.

The Minuend is a number that is to be separated into two parts, one of which is given for the purpose of finding the other.

The Subtrahend is the given part of the minuend.

The Remainder is the required part of the minuend.

The operation consists in the separation of the minuend into subtrahend and remainder. If they be re-united of course the minuend will result from the operation.

The simplest form of a problem in subtraction is that in which each term of the minuend equals or exceeds the corresponding term of the subtrahend. When the problem is presented in this form it is, of course, immaterial where the subtraction begins; e. g: 8256—4132= ? Problems rarely present this form however, yet all must assume it before the subtraction can be performed. There are two methods of changing the ordinary problems into the simplest form. The simpler is called the "changed minuend"

method, and consists in so changing the form of the minuend, without altering its value, that each of its terms shall equal or exceed the corresponding term of the subtrahend. In ordinary work the change is made as the necessity arises, but with beginners the wiser plan is to make all the changes at the same time. Take the following as an illustration: 7253—2879= ? Teach the pupils that 7253 equals 6 thousands, 11 hundreds, 14 tens and 13 units.. The problem then assumes the following form:

6	11	14	13
2	8	7	9

When this is understood let the changes in the minuend be made as the necessity arises.

The explanation will assume a form sustantially like the following:

"Since I cannot separate 3 units into two parts, one of which is 9 units, I reduce 1 ten to units: 1 ten = 10 units; 10 units + 3 units = 13 units. If 13 units be separated into two parts, one of which is 9 units, the remainder will be 4 units, etc."

CAUTIONS.

1. Do not permit the pupils to "borrow." "Borrowing" with no purpose of returning is a bad habit.

2. Where analyses like the above are written, allow no false statements of equality. Forms like the following are common: 1 ten = 10 units + 3 units = 13 units — 9 units = 4 units. From the foregoing we must conclude that 1 ten = 4 units. Require the writing to be truthful, even at the expense of time and material. 1 ten = 10 units. 10 units+3 units=13 units. 13 units—9 units=4 units. To prevent false forms in writing, insist that the oral analysis shall be equally explicit.

3. To furnish additional tests, assign a written analysis for home work. Insist that the papers shall be uniform in size and folding, that they shall be written with ink and that they shall be neat.

A second method of changing a problem into the form for subtraction is called "The Old Method." It rests upon the axiom that if two numbers be equally increased their difference remains the same.

Illustration: 7253 — 2879 = ?

Since I cannot separate 3 units into two parts, one of which is 9 units, I add 10 units to the units' term of the minuend. 3 units + 10 units = 13 units. 13 units — 9 units = 4 units. Since I have added 10 units to the minuend I must add 10 units to the subtrahend. (Why?) 10 units = 1 ten. 7 tens + 1 ten = 8 tens, etc.

CHAPTER IV.

As has been said, Multiplication is a method of uniting numbers. It may be regarded as a special case in addition, in which the numbers to be united are equal. $3 \times 4 = ?$ simply means, what is the sum of three fours? The facts of the multiplication table are fixed in the memory, and the operation consists in appealing to these memories.

DEFINITIONS.

1. Multiplication is a short method of uniting equal numbers.

2. The Multiplicand is one of two or more equal numbers that are to be united.

3. The Multiplier is the number of equal numbers that are to be united.

4. The Product is the sum of two or more equal numbers that have been united.

An understanding of the literal meaning of the above terms will be helpful.

From the definition of number, it follows that the distinction ordinarily made between *Abstract* and *Concrete* numbers has no foundation in fact.

Although number is purely abstract, a knowledge of it can be acquired only by the use of objects; hence the terms may be found convenient; with this in mind, we may enunciate the following

PRINCIPLES.

1. The Multiplicand may be either abstract or concrete.

2. The Multiplier is always abstract.

3. The Product is like the Multiplicand.

That many pupils who are studying written arithmetic have no clear idea of the nature of multiplication is demonstrated by the fact that when asked to show with objects, the meaning of 3×5, they will take three objects in one hand and five in the other. 3×5 does not mean *three fives* to them. They have studied figures, not numbers. The use of "times" is, perhaps, responsible, in part, for such a mental condition. The word is never a necessity. It is some-

times a convenience, but in many cases, probably, serves to hide knowledge. It is wiser to omit it until the nature of multiplication is understood.

LANGUAGE WORK.

Do not under-estimate the value of the language drill. Insist upon clear, smooth and grammatical explanations. A form similar to the following may profitably be insisted upon:

(1.) $489 \times 364 = ?$

$489 \times 364 = 9 \times 364 + 80 \times 364 + 400 \times 364.$

9×4 units $= 36$ units $= 3$ tens and 6 units.

9×6 tens $= 54$ tens. 54 tens $+ 3$ tens $= 57$ tens $=$ 5 hundreds and 7 tens, etc.

$80 \times 364 = 8 \times 10 \times 364.$ $10 \times 364 = 364$ tens.

8×4 tens $= 32$ tens, etc. $400 \times 364 = 4 \times 100 \times 364.$ $100 \times 364 = 364$ hundreds.

4×4 hundreds $= 16$ hundreds, etc.

The above form will be found simpler than the more common form.

(2.) Pupils often blunder in explaining the multiplication by the factors of a number. This should not be permitted. $24 \times 36 = 6 \times 4 \times 36,$ $4 \times 36 = 144.$ $6 \times 144 = 864.$

(3,) Pupils should acquire facility in analyzing problems involving concrete numbers.

What is the cost of 80 acres of land at $24 an acre?—Analysis: If each acre cost $24, 80 acres cost 80 × $24, or $1920.

Why not 24 × 80? Why is *each* better than *one?*

(4.) A very useful exercise, and one that prepares pupils for future work, and emphasizes the principles of multiplication, consists in changing the order of factors and explaining carefully.

For example, make 24 the multiplier in the above problem.

It is evident from the third principle that there can be no product which shall be dollars until we have a multiplicand that is dollars. Analysis: If each acre cost but one dollar, 80 acres would cost 80 dollars. Since each acre cost 24 dollars, 80 acres cost 24 × 80 dollars, or 1920 dollars.

The ability to make a supposition and show what the result would be, and then so modify the result as to make it agree with given conditions, is very valuable, not only in arithmetic, but in all subsequent mathematical study.

(5.) "To multiply by 10, 100, 1000, etc., annex as many zeroes to the multiplicand as there are in the multiplier."

The above statement is a very common one. Is it correct? If I annex one zero to the right of 78. I have 78.0. Have I multiplied 78 by 10?

It may be said that usually the decimal point is not placed at the right of an integer. But suppose the number had been 7.8, we should then have had 7.80. Is it not a better statement to say: To multiply by 10 make the number stand one order farther to the left, or (if the decimal point is used), remove the decimal point one order to the right?

(6.) There are several short methods of multiplication, and with some of these, at least, pupils should become familiar. They serve a double purpose. They are of considerable practical importance, and the analysis of them prepares the way for explanations of processes in fractions.

In this list should be included multiplication by aliquot parts of one hundred. The pupils should learn the following "aliquots":

One-half; the thirds; the fourths; one-sixth and five-sixths; one, three, five and seven-eighths. The twelfths and sixteenths may be added where time permits.

ILLUSTRATION.

Multiply 864 by 37½.

Form:

$$\begin{array}{r} 864\times 37\tfrac{1}{2} \\ \hline 86400 \\ 10800 \\ 32400 \end{array}$$

Insist upon *a particular form*, promptness, and neatness.

Explanation. 37½=⅜ of 100.

Were the multiplier 100, the product would be 86400, obtained by moving the multiplicand two orders to the left. Were the multiplier one-eighth of 100, the product would be one-eighth of 86400, or 10800. Since the multiplier is three-eighths of 100, the product is three times 10800, or 32400.

Require the pupils to write analyses similar to the above, giving especial attention to paragraphing, capitals, spelling and punctuation.

(7.) An extension of this work is advisable where circumstances are favorable, thus: 37½=37 5. 375 of any kind is three-eighths of one of the first order at the left of the 3. 3.75=⅜ of 10; .375=⅜ of 1;

375 = ⅜ of 1000, etc. In the same way, read 8.3⅓. .833⅓; 1.875: 31250.

Example. Multiply 482 by 6875.

Form.

$$\begin{array}{r} 482 \times 6875 \\ \hline 4820000 \\ 301250 \\ \hline 3313750 \end{array}$$

Explanation. $6875 = \frac{11}{16}$ of 10000.

Were the multiplier 10000, the product would be 4,820,000, obtained by making the number stand 4 orders farther to the left. Were the multiplier 1-16 of 10000, the product would be 1-16 of 4,820,000, or 301,250. Since the multiplier is 11-16 of 10000, the product is $11 \times 301{,}250$, or 3,313,750.

The methods above noted form a large part of the "stock-in-trade" of many "lightning calculators."

(8.) Many text-books give a short method for multipliers which are near a power of ten.

Multiply 87964 by 997.

Form.

$$\begin{array}{r} 87964 \times 997 \\ \hline 87964000 \\ 263892 \\ \hline 87{,}700{,}108 \end{array}$$

Explanation. 997 is 3 less than 1000.

Were the multiplier 1000, the product would be 87,964,000, obtained by making the number stand three orders farther to the left. Since the multiplier is 3 less than 1000, the product is $3 \times 87,964$ less than 87,964,000.

To be able to use the sixteenths of one hundred readily as multipliers, pupils must learn the first nine multiples of sixteen.

Give drill exercises upon them until they are mastered. Since in this work sixteen is to be used as a divisor, the following exercise will be found advantageous. Write a large number on the board, as for example,

258938756983.

Let the pupil name (1) the largest multiple of sixteen in the partial dividend, then (2) the quotient, next (3) the remainder, and finally (4) the new partial dividend.

In the above example the numbers spoken would be (1) sixteen; (2) one; (3) nine; (4) ninety-eight. The process would then be repeated until the close. When facility is required only the successive terms of the quotient should be spoken, the other results being *thought* only.

CHAPTER V.

Division is by far the most difficult of the "fundamental" processes. This is due to the fact that two processes have received the same name. $18 \div 3$ is usually read, "Divide 18 by 3," yet it has been translated "Separate 18 into threes," and "Find one-third of 18, that is, separate 18 into three equal parts."

The processes with objects are far from identical. A bundle of sticks is separated into threes by counting off a bundle of three, and continuing the operation until the bundle is separated into smaller bundles of three each. The second operation consists in starting three smaller bundles with one in each, and increasing them equally until the operation is completed. In the first case there are found to be 6 bundles of three each; in the second, each of the three is found to contain 6 sticks.

It seems wiser to have a name for each operation. If the term Division be reserved for the former, the

word Partition may be appropriately used for the latter. Perform the operation with objects and test the accuracy of the following

DEFINITIONS.

Division is the process of separating a number into equal numbers of a given size for the purpose of finding the number of such numbers in the given number.

The Dividend in Division is the number to be separated into equal numbers of a given size.

The Divisor in Division is the number showing the size of the equal numbers into which the Dividend is to be separated.

The Quotient in Division, is the number which shows the number of equal numbers into which the Dividend has been separated.

The Divisor is like the Dividend. The Quotient is abstract.

In "proving" Division, which becomes the multiplier? Why?

Partition is the process of separating a number into a given number of equal numbers for the purpose of ascertaining the size of each.

The Dividend in Partition is the number which is to be separated into a given number of equal numbers.

The Divisor in Partition is the number which shows the number of equal numbers into which the Dividend is to be separated.

The Quotient in Partition is the number which shows the size of the equal numbers into which the Dividend has been separated.

In "proving" a problem in Partition which becomes the multiplier? Why?

An example in Division:

If 15 apples be divided into piles of three apples each, how many piles will there be?

Analysis: If 15 apples be divided into piles of three apples each there will be as many piles as there are threes of apples in fifteen apples. There are five threes of apples in fifteen apples: hence there will be five piles.

An example in Partition:

If 15 apples be divided into 5 equal piles, how many apples will there be in each pile?

Analysis: If 15 apples be divided into 5 equal piles, there will be one-fifth of 15 apples, or 3 apples, in each pile.

The following is a briefer statement of the subject:

Division and Partition are processes of finding a second factor when the product and one factor are given. The Divisor is the given factor. The Dividend is the given product. The Quotient is the required factor. When the given factor and the product are alike the problem falls under Division; when unlike, under Partition. It is wiser to confine this statement to problems in which the quotient is integral. When, in performing a problem, the several products and remainders are written, the division is said to be *Long;* when these are not written the division is called *Short.*

METHODS OF EXPLAINING ABSTRACT PROBLEMS.

Division. $8634 \div 9$ (Read, divide 8634 into nines.) There are 9 nines in 86, with a remainder of 5. Since the 86 is hundreds the quotient and remainder are hundreds. 5 hundreds = 50 tens. 50 tens + 3 tens are 53 tens. There are 5 nines in 53, with a remainder of 8. Since the 53 is tens, the quotient and remainder are tens. 8 tens = 80 units. 80 units + 4 units = 84 units. There are 9 nines in 84, with

a remainder of 3; hence in 8634 there are 959 nines with a remainder of 3.

Partition. $8634 \div 9$ (Read, find $\frac{1}{9}$ of 8634.)

$\frac{1}{9}$ of 86 hundreds = 9 hundreds, with a remainder of 5 hundreds. 5 hundreds = 50 tens. 50 tens + 3 tens = 53 tens. $\frac{1}{9}$ of 53 tens = 5 tens, with a remainder of 8 tens. 8 tens = 80 units. 80 units + 4 units = 84 units. $\frac{1}{9}$ of 84 units = 9 units, with a remainder of 9 units.

Require each concrete problem to be read promptly and accurately, assigned to its proper case and analyzed. By this time the pupils should have acquired the ability to talk with decided fluency. Hesitation and stumbling should not be tolerated.

Changing problems from one case to the other will be found to be a profitable exercise

ILLUSTRATION.

A bought a farm at $25 an acre, paying $2875 for it; how many acres did it contain?

This is a problem in Division, because Divisor and Dividend are like numbers.

It may be changed to Partition.

Had he paid but one dollar an acre, $2875 would have bought 2875 acres. Since he paid $25 an acre, $2875 bought only $\frac{1}{25}$ of 2875 acres, or 115 acres.

A bought 115 acres of land for $2875, what was the price an acre?

This is a problem in Partition. Why?

It may be changed to Division.

Had he paid only one dollar an acre, 115 acres would have cost $115. Since the 115 acres cost $2875, he paid as many dollars an acre as there are 115's of dollars in $2875. There are 25 115's of dollars in $2875; hence he paid $25 an acre.

Division by aliquot parts of 100, 1000, etc., furnishes a good opportunity for language drill, fits the pupils for other analyses, and supplies a short method of obtaining results in many problems

ILLUSTRATION.

Problem.—$86485 \div 87\frac{1}{2} = ?$

Form.

$86485 \div 87\frac{1}{2}$

864.85

6918.80

988.4

Explanation. $87\frac{1}{2} = \frac{7}{8}$ of 100.

Were the divisor 100, the quotient would be 864.85, obtained by removing the number two orders to the right. Were the divisor $\frac{1}{8}$ of 100, the quotient would be 8×864.85, or 6918.8. Since the divisor is $\frac{7}{8}$ of 100, the quotient is $\frac{1}{7}$ of 6918.8, or 988.4.

Do not overlook what may be called the "elocution" of an oral analysis.

Pronunciation should be distinct and accurate, and contrasted terms should be properly emphasized. Pupils should stand erect and should face the teacher or the class instead of the blackboard.

It is convenient, often, to simplify a problem by dividing by the factors of the divisor.

The explanation should be given with care. $\frac{1}{24}$ of $576 = \frac{1}{4}$ of $\frac{1}{6}$ of 576. $\frac{1}{6}$ of $576 = 96$. $\frac{1}{4}$ of $96 = 24$.

The process of finding the true remainder is often puzzling to pupils.

Example. $8661 \div 42 = ?$

Into how many groups of 42 units each can I separate 8661?

8661 ones = 4330 twos, with a remainder of one one. 4330 twos = 1443 sixes, with a remainder of

one two. 1443 sixes=206 forty-twos, with a remainder of one six; hence, 8661 ones equals 204 forty-twos, with a remainder of one six, one two and one one, or nine.

Short methods of dividing by numbers that lack but little of a power of ten, are given in many text-books. They are of little value aside from the drill afforded in mastering them.

Example. $8967843 \div 98 = ?$

Form.

Quotients.	Remainders.
89678	43
1793	56
35	86
	70
2	55
	4
91508	59

Explanation. 100 is contained in 8976843, 89678 times, with a remainder of 43. 98 is contained 89678 times, with a remainder of 43, and a further remainder of two for each time 100 is contained. This further remainder is 179356. Proceeding as above with this number the first remainder is 56 and the further remainder is 3586. Proceeding in the same

way with 3586, the first remainder is 86 and the further remainder is 70, a number less than the divisor. The sum of the first remainders is 255. Proceeding as above with 255 the first remainder is 55 and the further remainder is 4. The sum of the quotients is 91508, and of the remainders is 59.

CHAPTER VI.

FACTORING.

If pupils are to become expert in arithmetical operations they must learn to factor numbers with celerity. Especially is this the case in Least Common Multiple, Greatest Common Divisor, Fractions, Percentage and its applications, and Proportion. A pupil who has had thorough drill in factoring will perform nineteen-twentieths of the problems in Fractions, found in the average written arithmetic, without the aid of his pencil; and few exercises do so much toward giving power to a class as drill of that kind.

As introductory to the work the pupils should memorize the factors of numbers to one hundred. These should be, like the multiplication table, part of their stock in trade. They should be fixed in the memory so that they will come immediately when called for.

In testing a class give dictation board exercises, with all the pupils at the board, if possible, and so separated that they shall not help each other. This may be done, as has already been suggested, by letting them number, and by giving one kind of work to the even numbers and another to the odd.

The board work should be scrupulously neat. The following form is suggested:

$$28 = 2 \times 2 \times 7.$$
$$36 = 2 \times 2 \times 3 \times 3.$$
$$48 = 2 \times 2 \times 2 \times 2 \times 3.$$

The pupil may read the above work as an equation, or as follows:

The prime factors of 28 are two twos and seven; of 36 are two twos and two threes; of 48 are four twos and three.

While it is true that $28 = 2 \times 2 \times 7$, it is not true that the prime factors of 28 are 2 times 2 times 7, as many pupils recite; much less is it true of "two square, seven."

In this connection teach the definitions of Prime, Composite, Factor, Common Factor, Prime Factor, Prime to each other, Odd, Even, Multiple, Common

Multiple. The etymologies of these terms will help to a better appreciation of their meaning. The unabridged dictionary will furnish them, and the pupils should form the habit of consulting its pages for that kind of aid.

TESTS OF DIVISIBILITY.

1. All even numbers are divisible by two.

2. A number is divisible by three if the sum of the ones expressed by its digits is divisible by three.

3. A number is divisible by four if the number expressed by the two right-hand digits is divisible by four.

4. A number is divisible by five if the right hand digit is zero or five.

5. A number is divisible by six if divisible by two and three.

6. A number is divisible by eight if the number expressed by the three right-hand digits is divisible by eight.

7. A number is divisible by nine if the sum of the ones expressed by the digits is divisible by nine.

8. A number is divisible by ten if the right-hand figure is zero.

9 A number is divisible by eleven if the sum of the ones expressed by the digits in the odd orders equals the sum of the ones expressed by the digits in the even orders, or if the difference of these sums is a multiple of eleven.

Other tests might be made by combining these, but enough have been given to answer ordinary purposes.

Pupils are troubled often in determining whether a number is prime or composite. They continue the division until they think there is reasonable ground for considering it prime, simply because the divisor is large. They should not be in doubt.

A number is prime if successive primes have been tried unsuccessfully until the integral part of the quotient is less than the divisor last used.

Since no smaller number than the prime last used will divide it, no larger number will, for if it were possible for a larger number to divide it the quotient would be a smaller number; but the quotient is a factor of the dividend, and we have seen that the dividend has no factor smaller than the prime used last.

Principles Employed in Factoring.

1. A divisor of a number is a divisor of any multiple of that number.

This is evident since every time the number is repeated the divisor is repeated.

2. A divisor of two numbers is a divisor of their sum.

Each is some number of times the common divisor, hence their sum is some number of times the common divisor.

3. A divisor of two numbers is a divisor of their difference.

Since each is some number of times the common divisor, if they differ it must be because one contains the common divisor more times than the other, hence the difference is some number of times the common divisor.

Demonstration of Tests.

test for four.

1. Any number of more than two orders may be regarded as some number of hundreds plus the number expressed by the two right-hand figures.

Since one hundred is divisible by four, any number of hundreds must be by Principle 1. If the number expressed by the two right-hand figures is divisible by four the whole number is by Principle 2.

TEST FOR FIVE.

2. Any number of more than one order may be regarded as some number of tens plus the number expressed by the right-hand figure. If the right-hand figure is zero the number is tens, and since one ten is divisible by five the number must be divisible by five by Principle 2.

TEST FOR EIGHT.

3. Any number of more than three orders may be regarded as some number of thousands plus the number expressed by the three right-hand figures. Since one thousand is divisible by eight, any number of thousands is so divisible by Principle 1. If the number expressed by the three right-hand digits is divisible by eight the whole number must be by Principle 2

TEST FOR NINE.

4. Any number may be separated into two parts, one of which is a multiple of nine and the other the sum of the ones expressed by the digits. Since by definition the first part is divisible by nine, it follows that, if the second part is, the whole number is, by Principle 2.

The first statement in the above demonstration may need elaborating somewhat.

One ten is one more than nine units; that is, it is one more than a multiple of nine. Two tens are two more than a multiple of nine. Any number of tens is as many more than a multiple of nine as there are tens. The same may be said of one hundred, two hundreds, or any number of hundreds; so of thousands, ten thousands, etc.

Or, making a general statement, since, in a decimal system, any unit equals ten of the next lower order, it must follow that any decimal unit is one of the next lower order more than a multiple of nine; hence one in any order expresses a number that is one more than the multiple of nine; two, a number that is two more than a multiple of nine; any digit, a number as many more than a multiple of

nine as the number of ones expressed by the digit. If the sum of these several remainders is a multiple of nine, the number must be.

(Frame a demonstration of the test for three from the above.)

Pupils should be able to apply these tests rapidly. Statements similar to the following will be found convenient:

Factor 25864.

This number is divisible by 8 because 864 is so divisible. Removing this factor the quotient is 3233. This number is odd, hence is not divisible by any even number. Since the sum of the ones expressed by the digits is 11 the number is not divisible by 3 or any of its multiples, because, etc. On trial I find it is not divisible by 7. It is not divisible by 11, because, etc., and in the same way the successive primes are tried to 53, which is found to give a quotient of 61. The prime factors then are 2, 2, 2, 53, 61.

If 3233 were prime, how long should the trials have continued? Why?

The explanation of the test for divisibility by eleven will be understood if the following facts are observed:

1. One ten is one less than a multiple of eleven; two tens are two less, and any number of tens are as many less as there are tens.

A similar statement may be made for thousands, hundred thousands, ten millions, etc. But tens, thousands, hundred thousands, ten millions, etc., occupy orders whose numbers, counting from the right, are even. Hence a digit standing in an order whose number is even expresses a number which is as many less than a multiple of eleven as the number of ones expressed by the digit.

2. In a similar manner it may be shown that a digit standing in an order whose number is odd expresses a number which is as many more than a multiple of eleven as there are ones expressed by the digit.

3. If these remainders are equal they balance each other and the number is a multiple. If one set of remainders exceeds the other by a multiple of eleven it must follow that the whole number is a multiple of eleven.

Although the tests of divisibility are indispensable aids to rapid work, it requires considerable drill to render pupils expert in their use. Rapid dictation exercises can be used advantageously to secure this

result. Send all the pupils to the board, or see that all are supplied with paper or slates.

AN EXERCISE IN THE THREE TEST.

The teacher pronounces the following numbers as rapidly as the pupils can work: 2841; 52689; 48263; 2875356; 796; 87234; and so following.

The pupils write:

2841,	15,	yes.
52689,	30,	yes.
48263,	23,	no.
2875356,	36,	yes.
796,	22,	no.
87234,	24,	yes.

The entire class can be tested in the time required for one by the oral method. The exercise tests the ability to write, add and divide. The work should be very neat, and the teacher should see the results.

These exercises should be continued until the pupils can apply the tests promptly and accurately.

CHAPTER VII.

CANCELLATION.

Cancellation is a method of shortening the work in problems involving only multiplication and division.

PRINCIPLES.

1. Dividing any one of a series of factors by any number divides their product by that number.

2. Dividing dividend and divisor by the same number does not change the quotient.

The first principle seems very simple, yet most pupils trip in its application. They do not recognize $24\times36\times75$, for example, as a number. This they should be accustomed to do,—a number that is partially factored. If a teacher should ask his class how he can divide such a number by three, many will answer "Divide each factor by three." Show the error. Read the number as 24 times 36 times 75.

One third of it is 8 times 36 times 75, 24 times 12 times 75, or 24 times 36 times 25.

The effect of the presence of o in a list of factors is often misunderstood. Try your classes on the following: $8\times6\times0\times4=?$ Some will answer "48;" others "4;" fewer will answer "nothing," as they should. Make it clear that whenever o enters as a factor the product is o.

$$\frac{24\times45\times60}{18\times20}=?$$

EXPLANATION.

This is a problem in division in which divisor and dividend are partially factored. I shorten the operation by dividing divisor and dividend by 9. One-ninth of the divisor is two times twenty. One-ninth of the dividend is twenty-four times five times sixty. I further shorten the operation by dividing divisor and dividend by four. One-fourth of the dividend is six times five times sixty, etc.

Pupils will soon be able to omit larger factors. Thus: one-eighteenth of the divisor is twenty. One-eighteenth of the dividend is twelve times five times sixty.

When the language work is well advanced pupils

should perform many problems without explanation, getting the result in the shortest time possible. Do not allow pupils to carry dead-weight in the shape of common factors. Encourage pupils to find short processes.

GREATEST COMMON FACTOR.

A factor of a number is a divisor of the number, or a number that is contained in it an integral number of times.

A common factor of two or more numbers is a divisor of each of them.

The greatest common factor of two or more numbers is the greatest number that will divide each of them.

A prime factor of a number is a divisor that is a prime number.

Any number is divisible by the product of two or more of its prime factors.

Pupils should learn to recognize all the divisors, whether prime or composite, of numbers.

EXERCISES.

1. Name all the prime divisors of 30, 39, 42, 50, etc.

2. Name all the divisors of the same numbers.

3. Name all the common prime divisors of 40 and 65; of 32 and 48; of 30, 45, 60, etc.

4. Review (3) naming all the common divisors in each case.

5. Review (4) naming the greatest common factor in each case.

PROBLEM.

What is the greatest common factor of 40, 48, 56?

FORM.

$$\left.\begin{array}{l} 40=2\times2\times2\times5 \\ 48=2\times2\times2\times2\times3 \\ 56=2\times2\times2\times7 \end{array}\right\} 2\times2\times2=8$$

EXPLANATION.

Two is a prime factor of each of the numbers, hence a factor of the greatest common factor. Another two is a prime factor of each of the numbers, hence a factor of the greatest common factor. A third two is a prime factor of each of the numbers, hence a factor of the greatest common factor.

Since there are no other common prime factors the product of two, two and two, or eight, is the greatest common factor. When the method of building the g. c. f. is mastered the explanation may be shortened.

Give many problems in which only the result is asked for in order to secure facility.

Observe that the g. c. f. is the product of the common prime factors.

Now drill the pupils in finding the g. c. f. by inspection. This can be done with readiness when the numbers are not large.

It is well for the pupils to use the words *factor* and *divisor* interchangeably.

An examination of the difference of two numbers or of the difference between one of them and some number of times the other, will often disclose their g. c. f. and save time.

What is the g. c. f. of 2373 and 2499? Their difference is 126; its prime factors are 2, 3, 3, 7. Since a common divisor of two numbers is a divisor of their difference, the g. c. f. of 2373 and 2499 must divide 126. Since a common divisor of two numbers is a divisor of their sum the g. c. f. of 2373 and 126 must be divisor of 2499 and must be the g. c. f. of 2373 and 2499, hence I examine only 126 and 2373.

Three is a factor of 2373, hence a factor of the g. c. f. Seven is a factor of 2373, hence it is also a factor of the g. c. f. Since these are the only common

prime factors their product is the g. c. f. of 2373 and 2499.

What is the g. c. f. of 387 and 2754? 2754 is 45 more than 7 times 387. Since a factor of a number is a factor of any multiple of that number, all the factors of 387 are factors of 7 times 387.

Seven times 387 and 2754 have no common prime factors that are not also common to 387 and 2754, hence by preceding principles the g. c. f. of 45 and 387 is the g. c. f. of 387 and 2754.

If the smaller number is a divisor of the larger it is the g. c. f. sought.

This method may be used with more than two numbers by finding the g. c. f. of the two, then of that number and a third, etc.

The methods that have been given of finding the greatest common factor of numbers are ample for all cases, but the method employed in algebra where quantities are given whose factors are not found by the application of the common theorems is presented in many arithmetics and may deserve discussion.

Its demonstration is too difficult for children and

should not be required unless the class is unusually mature.

Find the g. c. f. of 3139 and 4307.

The g. c. f. of these numbers cannot be greater than 3139. If 3139 will divide 4307 it is their g. c. f. It is contained once with a remainder of 1168, hence it is not their g. c. f. A common divisor of two numbers is a divisor of their difference, hence the g. c. f. of 3139 and 4307 must divide 1168; it consequently cannot be greater than 1168. If 1168 will divide 3139 it will also divide 4307, the sum of 1168 and 3139, and will be the g. c. f. sought. It is contained twice with a remainder of 803, hence it is not the g. c. f. of 3139 and 4307.

Since the g. c. f. must divide 1168 it must divide twice 1168, or 2336. (Why?) Since it must divide 2336 and 3139 it must divide their difference, or 803. (Why?) Hence it cannot be greater than 803. If 803 will divide 1168 it will divide 2336. (Why?) 3139, (Why?) and 4307, (Why?) and will be the divisor sought. It is contained in 1168 once with a remainder of 365, hence it is not the g. c. f. sought. Apply the same method of reasoning to 365.

It is contained in 803 twice with a remainder of 73. Apply the same method of reasoning to 73.

This method depends upon the three principles of factoring already given and is simply a substitution of smaller numbers which have the same g. c. f. as the numbers given.

CHAPTER VIII.

MULTIPLES.

A multiple of a number is an integral number of times that number.

A common multiple of two or more numbers is a multiple of each of them.

The least common multiple of two or more numbers is the least number that is a multiple of each of them.

What is the l. c. m. of 56, 72, 96.

FORM.

$$\left.\begin{array}{l} 56=2.\ 2.\ 2.\ 7, \\ 72=2.\ 2.\ 2.\ 3.\ 3, \\ 96=2.\ 2.\ 2.\ 2.\ 2.\ 3, \end{array}\right\} 2.\ 2.\ 2.\ 2.\ 2.3.3.7=2016=\text{l.c. m.}$$

EXPLANATION.

Since the l. c. m. of these numbers must contain 96 it must contain its prime factors, which I use as factors of the l. c. m. I wish to so change this pro-

duct that it shall also contain 72. This I do by introducing the additional factor 3. I now have a product that will contain 96 and 72. I wish to so change this product that it will also contain 56. This I do by introducing the additional factor 7. The l. c. m. is the product of five twos, two threes and seven. This number contains such prime factors and such only as are necessary to produce the several numbers.

The g. c. f. of these numbers is eight. The uncommon prime factors are 2, 2, 3, 3, 7. The l. c. m. of two or more numbers is the product of their g. c. f. and their uncommon prime factors.

ANOTHER METHOD.

When pupils have acquired considerable readiness in seeing the factors that compose a number, the work may be abridged very materially by employing what may be called the "inspection" method.

DIRECTIONS.

1. If any number is a factor of any other, strike it out, since its factors are contained in the other number.

2. Take the largest number and compare one of

the others with it, retaining the factors of the second not found in the first and striking out the others.

3. Compare a third with the first and remaining factors of the second, retaining factors found in neither and striking out the others and so proceed with the remaining numbers.

EXAMPLE.

Find the l. c. m. of 12, 18, 24, 27.

EXPLANATION.

Since 12 is a factor of 24, I strike it out. 8 contains all the prime factors of 24 not found in 27, hence I retain the 8 and strike out the 24. 18 has no factors not contained in 8 or 27, hence 8 times 27 is the l. c. m. of the numbers.

A THIRD METHOD.

Some problems include numbers not easily factored, and much time may be required in searching for divisors. The work may be shortened by taking two of the numbers, finding their g. c. f. by either of the last two methods, dividing one by the g. c. f. and multiplying the other by this quotient. Compare the multiple thus obtained with the third number and so following. This method depends upon the principle already stated. The l. c. m. of two num-

bers is the product of their g. c. f. and the uncommon prime factors, or, what is the same thing, it is the product of one of the numbers and all the prime factors of the other not contained in the first.

EXAMPLE.

Find the l. c. m. of 4087 and 4757.

EXPLANATION.

I find that the difference of these numbers is 670; its prime factors are 2, 5, 67. I see that neither 2, nor 5 is a factor of 4087. By trial I find that 67 is contained in 4087 sixty-one times; 67 is therefore a factor of 4757, for a factor of two numbers is a factor of their sum; moreover 67 is the only common factor. 61 times 4757 is, then, the l c. m. of the two numbers.

Remember the old proverb, "Practice makes perfect."

CHAPTER IX.

FRACTIONS.

Fractions contain little that is new. If the preceding work has been mastered, there should be but little difficulty. The new thing is the method of expressing the kind of units that compose the number.

A Fractional number is a number that is composed of one or more Fractional units.

A Fractional unit is a relative unit that is formed by separating an absolute unit into equal parts.

The expression $\frac{5}{6}$ is read *five sixths*. This is the only case in which 6 is read *sixths*. It is this fact, perhaps, which causes most trouble. The numerator of a Fractional number is the number of fractional units that compose the fractional number.

The Denominator of a Fractional number is the number of the fractional units required to make an

absolute unit. The Denominator may also be defined as the number of equal units into which an absolute unit has been separated. It thus indicates the size of the fractional units.

In the discussion of integers it was stated that there are two things to be considered in respect to each number; first, how many? second, what kind? In integers the first question is answered by the shape and order of succession of the figures; the second, by the position of the right-hand figure with respect to the decimal point. In fractional numbers the same condition exists. The numerator answers the first question, and the denominator the second.

We have, then, little or nothing that is new.

A Common Fraction is one whose numerator is written above and denominator below a short horizontal line.

A Decimal Fraction is one whose denominator expresses a power of ten.

The denominators of decimal fractions are not usually written; they are generally expressed by the position of the right-hand figure of the numerator with respect to the decimal point.

Now define Proper Fraction, Improper, Simple, Complex, Compound, and give and explain the Principles of Fractions given in all text-books.

As with integers, so with fractions, there are but two operations. They may be united or separated.

The Reduction of a number consists in changing its denomination without altering its value. The following reductions are performed in the study of Common Fractions:

I. A whole or mixed number to an improper fraction.

II. An improper fraction to a whole or mixed number.

III. A fraction to its lowest terms.

IV. A fraction to an equivalent fraction having any denominator.

V. Fractions to equivalent fractions having a common denominator.

I.

$7\frac{3}{8}$ is called a mixed number because there are two kinds of units, viz.: seven units of one kind and three of another. The reduction consists in changing this number to one in which there is but one kind of unit.

Since in one there are eight eighths, in seven there are seven eights of eighths, or fifty-six eighths. Fifty-six eighths plus three eighths equal fifty-nine eighths. Form a rule.

II.

An improper fraction is a simple number; that is, the units are of one kind. The reduction consists in changing it to another simple number (whole), or to a mixed number.

$\frac{27}{8} = ?$ Since in one there are eight eighths, in 27 eighths there are as many ones as there are eights in 27. There are $3\frac{3}{8}$ eights in 27: hence 27 eighths$=3\frac{3}{8}$ ones. Form a rule.

III.

A fraction is in its lowest terms when numerator and denominator are prime to each other, that is, when they have no common prime factors.

Since the g. c. f. is the product of the common prime factors it is clear that if the terms be divided by their g. c. f. the resulting quotients will be prime to each other.

$\frac{25}{30} = ?$ Dividing the terms by 5, their g. c. f., the resulting fraction equals $\frac{5}{6}$. It is obvious that $\frac{25}{30} =$

$\frac{5}{6}$, for although there are but $\frac{1}{5}$ as many fractional units, each is 5 times as large as before; or, considering the fraction as a problem in division we have the principle that dividing the divisor and dividend by the same number does not change the quotient. Form a rule.

IV.

Change $\frac{3}{7}$ to twenty-firsts.

Since in one there are $\frac{2}{2}\frac{1}{1}$, in $\frac{1}{7}$ there are $\frac{3}{2}\frac{}{1}$, and in $\frac{3}{7}$ there are three threes of twenty-firsts, which equal $\frac{9}{2}\frac{}{1}$.

Change $\frac{5}{9}$ to tenths.

Since in one there are $\frac{1}{1}\frac{0}{0}$, in $\frac{1}{9}$ there are $\frac{1}{1}\frac{1}{0}$, and in $\frac{5}{9}$ there are 5 times $\frac{1}{1}\frac{1}{0}$, which $= 5\frac{5}{1}\frac{}{0}$. Form a rule.

ANOTHER METHOD.

Since multiplying numerator and denominator by the same number does not change the value of the fraction, multiply both terms by such a number as will make the given denominator equal the required denominator. How shall this multiplier be found? Form a rule.

V.

In changing fractions to equivalent fractions having a common denominator, why do we select for the required denominator a *multiple* of the given denominators? Change $\frac{3}{4}$, $\frac{5}{6}$, $\frac{7}{10}$, to equivalent fractions having 15 for a denominator. What kind of fractions result? Why select the l. c. m. of the denominators?

The "Inspection method" of finding the l. c. m. will be found especially convenient. Use either method given in IV in making the reductions.

In making these several reductions teachers will find it advisable to employ object work, even with pupils of grammar grades. It is neither necessary nor expedient to use objects in all cases, as the pupils are mature enough to imagine them present. Require a clear statement of the exact process in I if the $7\frac{3}{8}$ were apples. Do the same in each case. III will be found quite difficult.

CHAPTER IX.

ADDITION OF FRACTIONS.

As in addition of integers, only like numbers can be added.

Like fractions are those which have the same fractional unit. Since the several numerators express the number of fractional units in the respective fractions, their aggregate expresses the sum of the fractional units in all of the fractions, hence the familiar rule.

Unlike fractions are those which do not have the same fractional unit. $\frac{2}{3}$ and $\frac{3}{4}$ cannot be united. Before addition is possible they must be made alike, or changed to equivalent fractions having a common denominator. In making this change encourage the use of the "inspection" method of finding the l. c. m. of the denominator.

Do no unnecessary work. Do not change mixed

numbers to improper fractions, and thus unnecessarily burden all of the subsequent processes.

Form a rule.

Encourage work like the following:

$\frac{3}{4}+\frac{5}{8}+1\frac{5}{16}+\frac{17}{20}=$? I wish to find how many ones there are in the sum of these fractions. $\frac{3}{4}$ lacks $\frac{1}{4}$ of being a unit. I make $\frac{1}{4}$ of $\frac{5}{20}$, thus leaving $\frac{12}{20}$ which equals $\frac{3}{5}$. $\frac{3}{4}+\frac{1}{4}=1$. $\frac{5}{8}$ lacks $\frac{3}{8}$ of being a unit; I make $\frac{3}{8}$ of $\frac{6}{16}$, leaving $\frac{9}{16}$. $\frac{5}{8}+\frac{3}{8}=1$. I am still to unite $\frac{9}{16}$ and $\frac{3}{5}$. $\frac{3}{5}$ lacks $\frac{2}{5}$ of being a unit. Since I cannot conveniently make fifths of sixteenths, I change $\frac{9}{16}$ to something of which fifths may be formed. $\frac{9}{16}=\frac{45}{80}$; $\frac{32}{80}$ are required to make $\frac{2}{5}$. $\frac{3}{5}+\frac{2}{5}=\frac{5}{5}$. $\frac{45}{80}-\frac{32}{80}=\frac{13}{80}$; hence, the result equals $3\frac{13}{80}$.

As in reductions, have the pupil think the process with objects.

SUBTRACTION OF FRACTIONS.

What is the fundamental principle of subtraction? If minuend and subtrahend are not alike what must be done with them?

An effort should be made to show the 'one-ness' of all subtractions.

$8\frac{5}{6}-4\frac{10}{11}=$?

Since $\frac{5}{7}$ and $\frac{1}{1}\frac{0}{7}$ are unlike, I change them to like numbers. How? $\frac{5}{7}=\frac{5}{7}\frac{5}{7}$. $\frac{1}{1}\frac{0}{7}=\frac{7}{7}\frac{0}{7}$. Since I cannot take $\frac{7}{7}\frac{0}{7}$ from $\frac{5}{7}\frac{5}{7}$, I take one unit from 8 units and reduce it to seventy-sevenths, etc. Form a rule.

An explanation similar to the following will be found helpful.

Separate $\frac{7}{8}$ into two parts, one of which is $\frac{2}{3}$. Since I cannot conveniently make thirds of eighths, I change $\frac{7}{8}$ to a fraction of which thirds may be made. $\frac{7}{8}=\frac{2}{2}\frac{1}{4}$. $\frac{1}{2}\frac{6}{4}$ are required to make $\frac{2}{3}$. If $\frac{2}{2}\frac{1}{4}$ be separated into two parts, one of which is $\frac{1}{2}\frac{6}{4}$, the other will be $2\frac{5}{4}$. Describe the process as it would be if performed with objects.

MULTIPLICATION OF FRACTIONS.

Review the definitions given in multiplication of integers. If the definition of multiplier be correct it is obvious that it cannot be a proper fraction.

$1\frac{8}{5}\times 6=?$

This problem may be read, "multiply $1\frac{8}{5}$ by 6," or, "unite 6 eights of fifteenths." 6 eights of fifteenths are $\frac{4}{1}\frac{8}{5}$ which =, etc.

The 6 groups, each containing $1\frac{8}{5}$ are reduced to one group containing $\frac{4}{1}\frac{8}{5}$. This in turn, is separated

into 3 groups, each containing $1\frac{5}{5}$, with a remainder of $1\frac{3}{5}$ or $\frac{1}{5}$, i. e., it is reduced to a mixed number.

The simplest method of multiplying a fraction by an integer, and the one most easily illustrated by the use of objects, is the multiplication of the numerator by the integer, since this gives the result obtained by uniting as many of the numerators as are expressed by the integer.

In such a problem as $\frac{7}{15} \times 5$, if the numerator be multiplied by 5, $3\frac{5}{15}$ will result. This will be reduced to lowest terms by dividing both terms by 5, hence three operations will have been performed, viz.: the multiplication of 7 by 5, the division of 35 by 5, and the division of 15 by 5. It is evident that the first two should be omitted, since the second simply undoes the first; hence in such cases no opportunity of dividing the denominator should be lost.

A second method of explaining the effect of dividing the denominator by the integer in the problem given above, is as follows:

By dividing the denominator of $\frac{7}{15}$ by 5 a fractional number results, each of whose units is equal to 5 of the former units. Since the number of units

is the same in the two cases it is obvious that the second number equals five of the first.

In the problem $\frac{8}{15} \times 6$, there is an opportunity to divide the denominator by a factor of the integer.

The formal statement is substantially as follows: I first multiply $\frac{8}{15}$ by 3 by dividing its denominator by 3, giving $\frac{8}{5}$, Multiplying $\frac{8}{5}$ by 2 the result is $\frac{16}{5}$. The more common form of expression is: Omitting the common factor 3, etc.

The following is a rule for multiplying a fraction by an integer:

a. Divide the denominator of the fraction by the integer if possible, or, if not possible,

b. Divide the denominator of the fraction by a factor of the integer, if possible, and multiply the numerator by the remaining factor, or, if not possible,

c. Multiply the numerator of the fraction by the integer.

With the definition of multiplication here employed such a problem as multiply 8 by $\frac{5}{6}$ is absurd. The problem should be read, find $\frac{5}{6}$ of 8. There is in such a problem a division (second case), and a multiplication. The multiplicand is $\frac{1}{6}$ of 8, and the multiplier is 5. Give an analysis and form a rule.

Such expressions as $\frac{5}{6} \times \frac{7}{8}$ should be read as $\frac{5}{6}$ of $\frac{7}{8}$. The habit of omitting common factors from numerator and denominator should be established.

Since such problems as $\frac{7}{15} \times \frac{9}{14} = ?$ involve both multiplication and partition, a full analysis of the process is deferred until Division has been discussed.

CHAPTER X.

DIVISION AND PARTITION OF FRACTIONS.

Review the definitions of Division and Partition.

The cases arising in this discussion are illustrated in the following problems:

(1) $8 \div \frac{3}{5} = ?$ (2) $\frac{4}{5} \div \frac{2}{7} = ?$

(3) $\frac{2}{7} \div \frac{4}{5} = ?$ (4) $1\frac{6}{5} \div 8 = ?$

(1) This problem is usually read, "Divide 8 by $\frac{3}{5}$." The meaning evidently is "Separate 8 into equal parts, each of which is $\frac{3}{5}$," or, "Find how many threes of fifths there are in 8."

In order that 8 may be separated into numbers, each of which is $\frac{3}{5}$, I change 8 to fifths. $8 = \frac{40}{5}$. In $\frac{40}{5}$ there are $13\frac{1}{3}$ threes of fifths.

This is the plan that would generally be followed if objects were employed to illustrate the problem. (What shorter plan with the objects?)

A rule based on the above would be as follows:

To divide an integer by a fraction, reduce the integer to the same denomination as the fraction, and divide the numerator of the dividend by the numerator of the divisor.

(2) $\frac{4}{5} \div \frac{2}{7} = ?$ How many twos of sevenths are there in $\frac{4}{5}$? This may also be read, "Separate $\frac{4}{5}$ into twos of sevenths."

In order that $\frac{4}{5}$ may be separated into two of sevenths, I change $\frac{4}{5}$ to a fraction of which sevenths may be made. $\frac{4}{5} = \frac{2}{3}\frac{8}{5}$. $\frac{10}{35}$ are needed to form $\frac{2}{7}$. In $\frac{2}{3}\frac{8}{5}$ there are $2\frac{4}{5}$ tens of thirty-fifths.

Hence, $\frac{4}{5} \div \frac{2}{7} = 2\frac{4}{5}$.

Describe the process if objects were employed.

Form a rule similar to (1).

A short explanation of Division by a fraction involves the following:

To divide one by a fraction, change one to the denomination of the fraction and divide the numerator of the dividend by the numerator of the divisor. In such a case, however, the numerator of the dividend will be the same as the denominator of the divisor. Hence, to divide one by a fraction, divide the denominator of the divisor by its numerator.

The following is a brief explanation of (2).

$\frac{4}{5} \div \frac{2}{7} = ?$

Were the dividend 1 the quotient would be $\frac{7}{2}$. Since the dividend is $\frac{4}{5}$ of 1 the quotient is $\frac{4}{5}$ of $\frac{7}{2}$.

$\frac{2}{7} \div \frac{4}{5} = ?$

Thoughtful pupils will observe that expressions of this character are not problems in division as we have used the term. Neither do they fall under Partition, as will shortly appear. They involve *Comparison.* The question really is, how far will $\frac{2}{7}$ go toward making $\frac{4}{5}$? or, $\frac{2}{7}$ is what part of $\frac{4}{5}$? $\frac{2}{7} = \frac{10}{35}$. $\frac{4}{5} = \frac{28}{35}$. $\frac{10}{35}$ is $\frac{1}{28}$, or $\frac{5}{14}$ of $\frac{28}{35}$.

They may, however, be regarded as problems in division, and may be analyzed according to the model given for division by a fraction.

(4) As division has been defined it is obviously impossible to divide $1\frac{6}{25}$ by 8, since $1\frac{6}{25}$ cannot be separated into numbers, each of which is 8. This is a problem in Partition. $1\frac{6}{25}$ may be separated into 8 equal parts. $1\frac{6}{25} \div 8$ should be read, find $\frac{1}{8}$ of $1\frac{6}{25}$, or, separate $1\frac{6}{25}$ into 8 equal parts.

Pupils should see what "Striking out" common factors means.

This has been shown in the multiplication of a fraction by an integer; let us observe the effect in problems like (4).

$\frac{1}{1}\frac{4}{7} \div 21 = ?$

Observing the factor 7 in 14 and 21, the problem may be read: Find $\frac{1}{3}$ of $\frac{1}{7}$ of $\frac{1}{1}\frac{4}{7}$. $\frac{1}{7}$ of $\frac{1}{1}\frac{4}{7} = \frac{2}{1}\frac{}{7}$. $\frac{1}{3}$ of $\frac{2}{1}\frac{}{7} = \frac{2}{5}\frac{}{1}$.

Why does multiplying the denominator of a fraction divide the fraction?

The denominator of a fraction shows the size of the fractional units which compose the fraction. This it does by showing the number of fractional units into which the absolute unit has been separated. If this number be increased the size of the fractional unit will be correspondingly diminished.

ANOTHER VIEW.

$\frac{5}{8} \div 3 = ?$

Since $\frac{1}{3}$ of 5, mixed number, I separate each eighth into three equal parts, and have $\frac{1}{2}\frac{5}{4}$. $\frac{1}{3}$ of $\frac{1}{2}\frac{5}{4}$ is $\frac{5}{2}\frac{}{4}$. The operation involves the multiplication of the numerator and denominator by three, and the division of the resulting numerator by three—three

operations, two of which may be omitted, since the second undoes the first.

We are now ready to analyze such problems as $\frac{3}{4}\frac{4}{5} \times 2\frac{7}{17} = ?$

What is $\frac{3}{4}\frac{4}{5}$ of $2\frac{7}{17}$? $\frac{1}{4}\frac{}{5}$ of $2\frac{7}{17} = \frac{1}{5}$ of $\frac{1}{9}$ of $\frac{2}{1}\frac{7}{7}$. $\frac{1}{9}$ of $\frac{2}{1}\frac{7}{7} = 1\frac{3}{7}$. $\frac{1}{5}$ of $1\frac{3}{7} = \frac{3}{8}\frac{}{5}$. $\frac{3}{4}\frac{4}{5}$ of $\frac{2}{1}\frac{7}{7} = 34 \times \frac{3}{8}\frac{}{5} = 2 \times 17 \times \frac{3}{8}\frac{}{5}$. $17 \times \frac{3}{8}\frac{}{5} = \frac{3}{5}$. $2 \times \frac{3}{5} = \frac{6}{5}$.

The short statement for the same is, omit equal factors from numerator and denominator, etc.

It is worth while, however, to have a pupil know whether he is multiplying or dividing when he omits factors.

COMPLEX FRACTIONS.

A complex fraction is usually defined as "a fraction which has a fraction in one or both of its terms." Are all so-called complex *fractions* really fractions? A fraction has been defined as an expression for one or more of the equal parts of a unit. The complex fraction whose numerator is three-and-a-half, and whose demominator is four, evidently falls under the definition, but what shall we say of those whose denominators are fractions? Can we conceive of a unit as being divided into $\frac{1}{4}$ equal parts? Evidently the idea is absurd.

If the numerator is equal to or greater than one, and the denominator is integral, the expression is a fraction. If the numerator is less than one, or if the denominator contains a fraction, the expression is simply a problem in division—fractional only in form.

It seems wiser, at first, to regard them simply as problems in divison, and to have them so explained. Pupils should be able to read them as fractions, however, thus: $6\frac{3}{4}—\frac{7}{12}$ may be read as "the complex fraction whose numerator is $6\frac{3}{4}$ and denominator $\frac{7}{12}$." It may be read also as a problem in division: $6\frac{3}{4} \div \frac{7}{12}$.

As problems in division they present nothing new. It is worth while, however, for pupils to understand that the laws of simple fractions are equally applicable to them; that is, numerator and denominator may be multiplied and divided by the same number without changing the value of the fraction, and when proficiency has been acquired in reducing them by the ordinary method, let them try the following:

(1.) $3\frac{1}{2}—4\frac{1}{2}$. Since $\frac{1}{2}$ is a common factor of the terms, divide each term by it. This problem illustrates a large class.

(2.) $\frac{7}{8}—\frac{5}{12}$. Multiply both terms by 24, the l. c. m. of the denominators. Multiplying the nu-

merator first by 8, the product is seven; multiplying 7 by 3, the product is 21.

Multiplying the denominator first by 12, the product is 5; multiplying this product by 2 (the remaining factor of 24), the product is 10.

Any complex fraction may be simplified very readily by this process. If one or both of the terms are mixed numbers, do not reduce them to improper fractions. Reduce $4\frac{1}{5}—5\frac{3}{10}$. I multiply both terms by 10, the l. c. m. of their denominations. 5 times the numerator is 21; 2 times 21=42, etc.

It should be remembered that there is only one way of making pupils quick and accurate workers; they must perform problems by the hundred. Send the class to the board so that their methods of work can be seen, then dictate problems and require rapid work. Permit no erasing. Let every mark remain. If errors are made, and discovered by the pupils, the previous work can be crossed and corrections made, but all that has been done should appear upon the board. If pupils realize that their errors are to face them, they will be more careful about making them. In a twenty-minute recitation each pupil can perform from ten to thirty such reductions as are given above.

CHAPTER XI.

THE METHODS OF FINDING THE PART WHICH ONE NUMBER IS OF ANOTHER.

The two questions A is how many times B? and A is what part of B? are in substance the same. The question usually assumes the first form if A is greater than B; if A is less than B the question usually assumes the second form. The answer to the second is always fractional, while the answer to the first may be either integral or mixed.

FORMS OF PROBLEMS.

1. 7 is what part of 10?

One is $_1{}^1\!/_{10}$ of 10, hence 7 must be $_1{}^7\!/_{10}$ of 10. (How many objects are needed to illustrate it?)

2. $\frac{2}{3}$ is what part of 4?

Since only like numbers can be compared, I change 4 to thirds. $4 = {}^1\!/_3{}^2$. The question now is, $\frac{2}{3}$ is what part of ${}^1\!/_3{}^2$? $\frac{2}{3}$ are $\frac{1}{6}$ of ${}^1\!/_3{}^2$. Show with objects.

3. $\frac{3}{4}$ is what part of $\frac{5}{6}$?

Since only like numbers can be compared, I make the fractions alike. $\frac{3}{4} = \frac{8}{12}$; $\frac{5}{6} = \frac{10}{12}$. The question now is, etc.

It will be seen that the same result is obtained in each case if the first number be divided by the second; hence the common rule:

'Divide the number expressing the part by the number of which it is a part.

These relations may be shown very easily by folding paper squares.

To illustrate (3), fold the square into sixths, and then to twenty-fourths. Show to how many of the twenty-fourths the $\frac{5}{6}$ and $\frac{3}{4}$ are severally equal, and then how far the latter will go toward making the former.

STRAIGHT LINE ANALYSIS.

Familiarity with this topic will enable pupils to perform most problems involving only multiplication and division in a small part of the time usually required.

Most of the problems in multiplication and division of fractions, many in the applications of percentage, almost all in simple and compound propor-

tion, and very many others admit of such a solution. Indeed, since Proportion belongs more properly to algebra, it is questionable whether we should attempt it in arithmetic, unless the class has abundant time or is quite mature.

DIRECTIONS.

1. Begin with a number which is of the denomination of the required result. Since the product is like the multiplicand, and since the divisions may all be of the second case, in which the quotient is like the dividend, each result will be like the number with which we begin.

2. Fractions should appear in neither term and should be put upon the straight line a term at a time.

If the number with which we begin is mixed, change it to an improper fraction.

PROBLEM.

If 27 men in 18 days of 10 hours each dig a ditch 180 rods long, 6 feet wide, and 3 feet deep, of 5 degrees of hardness, in how many days of nine hours each will 45 men dig a ditch 300 rods long, 8 feet wide and 4½ feet deep, of 7⅔ degrees of hardness?

ANALYSIS.

Since the question asks for a number of days we begin with 18 days. Drawing a horizontal line, "18 days" is written above the left end.

One man could perform the work in 27 times 18 days, which is expressed by writing 27 as a factor of the dividend. We now have an expression for the number of days required for one man to do the work if 27 men could do it in 18 days. Since 45 men are to do the work, they can accomplish it in one-forty-fifth of this number of days, which is expressed by writing 45 as a factor of the divisor. We now have an expression for the number of days required for 45 men to do the work if 28 men can do it in 18 days.

If these men worked only one hour a day, ten times as many days would be needed. This is expressed by writing 10 as a factor of the dividend. What is now expressed?

Since the 45 men are to work 9 hours a day only one ninth as many days are required, which is expressed by writing 9 as a factor of the divisor. What is now expressed?

The ditch is 180 rods long. Were it but one rod long, only one one-hundred-eightieth as many days

would be needed. This is expressed by writing 180 as a factor of the divisor.

Since the proposed ditch is to be 300 rods long, 300 times as many days are necessary, which is expressed by writing 300 as a factor of the dividend. (The pupil should be able to tell what is expressed at any step.) Were the ditch but one foot wide, 45 men could dig it in one-sixth this number of days,—expressed, etc. Since the ditch is to be eight ft. wide, eight times this number of days are needed,—expressed, etc.

Were the ditch but one foot deep, one-third this number of days would be required,—expressed, etc. Were the ditch only one-half a foot deep, only one-half of this number of days would be necessary. Since the ditch is to be nine-halves feet deep, nine times this number of days are needed, and so on through the solution. When the work is indicated it will assume the following form:

$$\frac{18\times27\times10\times300\times8\times9\times23}{45\times9\times180\times6\times3\times2\times5\times3}$$

All common factors should now be omitted.

The work is necessarily slow at first, but the teacher should persist until it is understood and can be

given with accuracy and rapidity. It is a perfect machine, but, unlike most machine processes, it cannot be used unless it is understood.

The problems usually given in Cancellation are simple enough to introduce the topic. When they are exhausted turn to Simple Proportion. These will be found to be more difficult. Follow these with problems from Compound Proportion, similar to the one given above.

While introducing the work, those who find it very difficult should be aided by questions, although they should not be too suggestive.

Emphasize the fact that in analysis "*we go around by the unit;*" that is, if we are told that 19 men can do a piece of work in 24 days, and are asked how many days will be required for 50 men to do the same work, we first find the number of days that one man will require to do the work.

CHAPTER XII.

DECIMAL FRACTIONS.

A decimal fraction is one whose denominator is a power of ten. The denominator instead of being written, as in the common fraction, is usually expressed by the position of the right-hand figure of the numerator with respect to the decimal point. In an early number of this series directions were given for writing them. These should be reviewed with care. The pupils should learn the number of any order at the right of the point so that it can be given as soon as the name is pronounced.

Frequent dictation exercises will be found useful in fixing these facts. Require statements similar to the following:

"I write 629 millionths by making 9 stand in the sixth order at the right of the decimal point." In order to write readily the pupil must see:

(1.) How many figures are necessary to express the numerator.

(2.) Where the right-hand figure must stand to make the number express the required denomination, and

(3.) How many zeros (if any) must precede the numerator.

REDUCTIONS.

1. To change a decimal fraction to a common fraction erase the decimal point, write the denominator, and reduce to lowest terms.

This exercise will afford an excellent opportunity to review complex fractions.

2. To change a common fraction to a decimal.

(*a*) Divide the numerator by the denominator, or

(*b*) Do anything to the fraction, that you have a right to do, that will make its denominator a power of ten; then erase the denominator and express it by the position of the decimal point.

The first of these methods should be very easy to the pupil. As has been said, a fraction may be regarded as a problem in division. Seven-sixteenths is one sixteenth of seven.

In simple division, when the first term of the div-

idend would not contain the divisor, it was reduced to the next lower denomination. Do the same with these problems.

ANALYSIS.

Change seven-sixteenths to a decimal fraction.

7=7.0. (Read 70 tenths.) 1-16 of 7.0=.4 with a remainder of .6. .6=.60. 1-16 of .60=.03 with a remainder of .12. .12=.120. 1-16 of .120=.007 with a remainder of .008. .008=.0080. 1-16 of .0080=.0005. Hence, 7-16=.4375.

Oblige the pupils to keep the decimal point in position all of the time until the reduction is thoroughly understood. Do not allow them to say "I annex zeros to the numerator and divide it by the denominator, etc."

Only those common fractions which in their lowest term have in the denominators the factors 2 or 5, or both, and no others, can be changed to "pure decimals."

The reason for this becomes clear when it is observed that with each reduction of the numerator only the factors 2 and 5 are introduced.

The second method is much briefer in many cases, but requires more skill in its use. Complex fractions

whose denominators are aliquot parts of some power of ten may be reduced at once by multiplying both terms by the quotient arising from dividing the power of ten by the "aliquot."

EXAMPLES.

(1). $\frac{3}{12\frac{1}{2}}$. Multiply both terms by 8.

(2). $\frac{18}{3\frac{1}{3}}$. Multiply both terms by 3.

(3). $\frac{15}{6\frac{1}{2}}$. Multiply both terms by $1\frac{3}{5}$

The number by which to multiply the terms of a common fraction may be told by inspecting the denominator.

Change $\frac{7}{16}$ to a decimal fraction.

The denominator is the product of four twos. To change this fraction to a decimal fraction the denominator must be made a power of ten. Since ten is the product of two and five, any power of ten is the product of an equal number of twos and fives; hence multiply both terms of the fraction by the fourth power of five.

If, when the fraction is in its lowest terms, other prime factors than two and five are found in the denominator, such factors must be paired with mixed numbers or fractions to produce a power of ten; hence

the numerator will be a mixed number and the fraction will not be a "pure" decimal.

ADDITION AND SUBTRACTION.

There is nothing new in addition and subtraction. The methods and explanations previously employed are applicable and should be used.

MULTIPLICATION.

Multiplication of decimals presents nothing new except the "pointing" of the product.

The explanations given in multiplication of common fractions need no modification for application to "decimals."

Such expressions as $8.62 \times .023 = ?$ should be read *.023* of $8.62 = ?$

ANALYSIS.

One-thousandth of $8.62 = .00862$, obtained by moving the decimal point three places to the left. .023 of $8.62 = 23$ times .00862.

Most arithmetics explain the philosophy of "pointing" by changing the "decimals" to common fractions, performing the multiplication and then changing the product back to the original form.

It may also be explained as follows:

When the multiplier is units (of the first order), the product is like the multiplicand (has as many "places"). When the multiplier is of a denomination lower than units, the product is the same part of the product obtained by considering the multiplier units that the given multiplier is of the multiplier so considered.

DIVISION.

In division of decimals the forms of explanation in common fractions may be employed.

In all such problems the divisor will be an integer or a "decimal." If it is an integer the problem falls under partition and should be so explained. If it is a "decimal" it will be of the same denomination as the dividend, or of a higher or lower denomination.

When of the same denomination the analysis is identical with that first given for the division of a fraction by a fraction. When of a lower denomination the dividend may be reduced to the same denomination and the same analysis be employed. When of a higher denomination it may be reduced to the same denomination as the dividend and compared with it; or,

In all such cases divisor and dividend may be multiplied by a number that will make the divisor integral. The explanation may then be given by division or partition, as the case may be. If the divisor is less than the dividend, it may be division; if greater, it will be partition.

The "pointing" may be explained briefly as follows:

Since the dividend corresponds to the product, and the divisor and quotient to the factors, it must follow that the quotient must have a number of "decimal places" which added to the number in the divisor will equal the number in the dividend; or,

If the divisor were units (of the first order) the quotient would be like the dividend. See Partition. If the divisor is of a denomination lower than units, the quotient is as many times the quotient obtained by considering the divisor units as the divisor so considered is times the given divisor.

Require the pupils to explain the problem carefully, considering the divisor units. He will see that the quotient is like the dividend. In making the correction for the given divisor he will see that the period is moved as many places to the right in the quotient as there are figures at the right of the point in the divisor.

CHAPTER XIII.

COMPOUND NUMBERS.

This subject contains more information that may be made interesting than any other topic in the ordinary arithmetic.

Too often the only attention it receives is the mechanical learning of tables, definitions and rules, and the more mechanical grinding out of results in performing problems. The definitions and tables must be learned, and there is no escape from a certain amount of drudgery; but much can be done to relieve the work of its most irksome feature.

A compound number is a concrete number which is composed of units of more than one denomination, but is reducible to a simple number.

Or, a compound number is a concrete number in which the scale is not ten.

The latter definition, of course, excludes Federal money.

Number has been defined as the measure of the relation between quantities.

Compound numbers are measures.

They measure
- 1. Money.
- 2. Extension.
  - *a.* In one direction, or length.
  - *b.* In two directions or surfaces.
  - *c.* In three directions, or volumes.
- 3. Weight.
- 4. Time.
- 5. Angles.
- 6. Miscellaneous.

Measuring is the process of finding how many times a quantity contains another quantity, which we call the unit of measure.

The result is a *number* and simply gives us definite ideas as to the amount of the thing measured, or, in other words, enables us to understand each other.

The first requisite for measuring is a unit.

MONEY is usually defined as a medium of exchange; that is, it is that instrument by means of which the various commodities of commerce pass from hand to hand. It furnishes a standard of value; it is a measure.

The exchange of one article of use or consumption for another is called barter. Pupils will see how inconvenient this method of exchange might be. A has a certain article, but prefers to have another. He must find a person who has what A desires, and who, also, prefers what A has. This method prevails among rude peoples, but the advantages of a money system soon become apparent to an advancing race, hence, when a nation reaches a certain grade of civilization, some kind of a medium of exchange is adopted, or in other words, money appears.

Teachers will find it profitable, in this connection, to read the article on "Money" in some standard cyclopedia.

A scale is the statement of the number of units in each order required to make one of the next higher.

FEDERAL MONEY.

What is the scale in U. S. money? The unit is the dollar. What is the origin of the word *dollar*? Explain the origin of the dollar-sign. What is a dime? What is the origin of the word? Answer the same questions for cent and mill.

What coins are issued by the government? What is the building called in which they are made? What it the weight of a silver dollar? What part of the coin is alloy? Why is alloy used? What metal is employed? Silver coins weigh how many times as much as gold coins of equal value? How much is an ounce of gold worth?

A cubic foot of gold weighs about 1200 pounds avoirdupois; how much is it worth?

How many kinds of paper money are there? What is the name of each kind? About how much of each kind is there? If all the money in this country were divided equally among the inhabitants about how much would each have?

Teachers must determine how much of this kind of work will be profitable.

REDUCTION.

Reduction is the process of changing the denomination of a number without changing its value.

There are two forms of reduction. Reduction Ascending is the process of changing a number to a number of the same value, but of a higher denomination.

In the same way define Reduction Descending.

Do not talk about changing a number *from a lower* to a higher denomination. Omit the words in Italics.

FORM OF ANALYSIS.

Change 428 cents to mills.

1. Since in 1 cent there are 10 mills, in 428 cents there are 428 tens of mills, or 4280 mills.

It will be seen that this method makes the larger number the multiplier, and is not always the most convenient.

2. Were there only 1 mill in a cent, in 428 cents there would be 428 mills. Since there are 10 mills in a cent, in 428 cents there are ten 428's of mills, or 4280 mills.

3. Since there are 10 mills in a cent, in any number of cents there are 10 mills for every cent; hence in 428 cents there are ten 428's of mills.

One form of analysis is sufficient for an ordinary class. If only one is used, the third will be found most convenient.

REDUCTION ASCENDING.

Change 4820 mills to cents.

1. Since in one cent there are 10 mills, in 4820 mills there are as many cents as there are tens in 4820. There are 482 tens in 4820; hence in 4820 mills there are 482 cents,

2. If there were only one mill in a cent, in 4820 mills there would be 4820 cents. Since there are 10 mills in a cent, in 4820 mills there are one-tenth of 4820 cents, or 482 cents.

3. Since there are 10 mills in a cent, in any number of mills there are one-tenth as many cents; hence in 4820 mills there are one-tenth of 4820 cents, or 482 cents.

These forms of analysis apply to all problems in Reduction of Compound Numbers.

There is no reason why a knowledge of the form of bills and accounts should not be acquired at this point. A few cents invested in journal paper will supply the needed material. Pupils have a fondness for anything that looks like business. The farmer boys should be able to put their fathers' accounts upon paper in a neat, accurate form. The exercise may be substituted for the regular work in penmanship occasionally, may be assigned as a home lesson, or may be prepared during some school hour. Insist upon neatness and accuracy. Many teachers are so anxious about the quantity of their work done that the quality is miserable. A little work thoroughly done is infinitely better than three times as much done carelessly.

ENGLISH MONEY.

Pupils should become accustomed to recite the scale, in any case, without the table or with it, and forward and backward. Thus, in English money, the scale is 4, 12, 20.

The unit is the pound sterling. A pound weight

of silver was anciently divided into 240 equal parts, called pence, hence the origin of the first part of the name. "These pence were called *esterling*, whence the name 'sterling.' This is supposed by some writers to have been derived originally from Easterlings, the popular name of traders from the Baltic and from Germany, who visited London in the middle ages, and some of whom were probably employed in coining. By others it is supposed to be a diminutive of *star*, and in some old writings it is written *starling*, the penny being so called from the star often stamped upon it."—*Am. Cyclopedia.*

The pound sterling is worth $4.8665 in U. S. money. It is represented by the coin called the sovereign and by the one-pound bank note.

What is the meaning of *farthing?* Why so-called? Pupils should learn the names of English coins, and the value of each. Which is purer, the English or U. S. coin? A given weight of English coin is worth more or less than an equal weight of U. S. coin? Why?

Teachers must use their own discretion respecting the number of national currencies they will require

their pupils to study. All will require a mastery of English money, and if time permits, French and German should be added.

CHAPTER XIV.

MEASURES OF LENGTH.

The unit of length is the yard. It was determined in Great Britain about 1760.

Few pupils have any idea of the care exercised in fixing this standard. Experiments were made at London with pendulums of different lengths until one was found that beat seconds, that is, that vibrated 86,400 times in an average solar day. This pendulum was enclosed in a vacuum to protect it from atmospheric currents, and was suspended at the sea-level in order that it might be as near the earth's center of gravity as possible in that latitude without going below the earth's surface. The length of this pendulum was separated into 391,393 equal parts, and 360,000 of them were taken for a yard. A foot is

one-third of this standard, and an inch (from a word meaning a *twelfth*) is a twelfth of a foot.

Many pupils will be interested in ascertaining the origin of the names of the different units. An unabridged dictionary will furnish the information needed in nearly all cases.

Pupils should thoroughly master the tables of long measure found in their arithmetics. Recite the scales without the names forward and backward.

In the reductions use the forms of analysis already given.

MEASURES OF SURFACES.

A unit of length having been selected, the surface units are easily determined. Define *surface*, *plane*, *rectangle*, *square*.

The area of a surface is the number which expresses the number of times the surface contains the unit.

What is a square foot? A square rod? There may be a difference between "a foot square" and "a square foot," but it can be a difference of shape only. The fact that many pupils who have finished the subject are bothered by such questions as: "What is the

difference between three feet square and three square feet?" shows that their ideas are not clear. The surfaces should be represented to the eye until the matter is thoroughly understood.

It is not the purpose to discuss the whole subject of Compound Numbers, but to touch those points with which pupils have most trouble.

A field is 200 rods long and 120 rods wide; what is its area?

Most arithmetics tell us to multiply the length by the breadth. A scene like the following is not unusual:

Teacher. What kind of a number is the multiplier?

Class. The multiplier is *always* abstract.

T. What kind of a number is the product?

C. The product is *always* like the multiplicand.

T. What is the product of 8 feet and 12 feet?

C. (*Unanimously*) 96 *square* feet.

T. And how is it that you have a *concrete* multiplier and a product *unlike* the multiplicand?

This question is usually followed by a period of profound silence, as pupils begin to realize how little real faith they have in principles which they recite with absolute accuracy.

Multiplying the length by the breadth never gave the area and never will. If the number of units in the length be considered an abstract number and be multiplied by the number of units in the breadth, also considered as an abstract number, the product will be an abstract number containing as many units as there are surface-units in the area. It is needless to say that the point is too fine; it isn't. One statement is false and the other true. Teachers who permit such propositions to pass unquestioned will accept equations like the following: $4 \times 2 = 8 + 4 \div 6 = 2$.

An analysis something like the following should be required:

If the field were a rod long and a rod wide, it would contain one square rod. If it were 200 rods long and 1 rod wide, it would contain 200 sq. rods. Since it is 200 rods long and 120 rods wide, it must contain 120 times 200 sq. rods, or, etc.

Show the relations of the tables of surface measure to the table of linear measure. In the reduction use the form of analysis given.

MEASURES OF VOLUME.

Define solid, cube, cubic inch, cubic foot, cubic yard, etc.

The unit used in measuring volumes is a cube whose edge is some linear unit, or some unit containing a specified number of these cubes.

Show the relations of the tables of "solid measure" to those of "long" and "square" measure.

ANALYSIS.

What is the volume of a rectangular parallelopiped whose length is 8 feet, width 4 feet, and thickness 3 feet?

If the figure were 1 foot long, 1 foot wide, and 1 foot thick, it would contain 1 cubic foot. If it were 8 feet long, 1 foot wide, and 1 foot thick, it would contain 8 cubic feet. If it were 8 feet long, 4 feet wide, and 1 foot thick, it would contain 4×8 cubic feet$=32$ cubic feet. Since it is 8 feet long, 4 feet wide, and 3 feet thick, it contains 3×32 cubic feet$=$ 96 cubic feet.

LUMBER MEASURE.

A foot of lumber contains 144 cubic inches. A board that is one inch thick and one inch wide, then, must contain one-twelfth of a foot of lumber for each foot of length. Twelve-foot boards contain how much lumber for each inch of width? Answer the

same question for boards of any length. Ordinary studs are 2 by 4 inches. How much lumber do they contain for each foot of length?

Teachers who can borrow a lumber dealer's measure will find it an object of interest to their pupils. It is furnished with several scales so graduated that the number of divisions which measure the width of a board expresses the number of feet in the board if it is one inch thick. What part of an inch should each division be for 16 feet lumber? Answer the same question for boards of any length.

LIQUID AND DRY MEASURE.

Pupils should see clearly that the gallon and bushel are measures of volume that contain a specified number of cubic inches. In liquid measure the gallon of 231 cubic inches in the unit, while in dry measure the Winchester bushel of 2150.42 cubic inches is the standard.

Find the number of cubic inches in the liquid quart, also the number in the dry quart. Why do they differ? What is the difference between the liquid gallon and the dry gallon? Why do they differ?

The apothecaries' fluid measure is often omitted,

yet the terms are in common use. Pupils should know what is meant by a two-ounce vial. But little time will be needed to master this measure, and bottles of various sizes should be brought to the recitation. Interest in the whole subject will be greatly increased by having the various measures at hand for the pupils to use.

The reductions are explained by the methods of analysis already given.

MEASURES OF WEIGHT.

It should be seen that all measures of extension thus far discussed, have been derived from the unit of length—the yard.

It should now be seen that all the measures of weight are derived from the same source.

The unit from which the various units of troy, apothecaries' and avoirdupois weight are derived is the "troy pound of the mint." But this unit is simply a piece of metal that weighs as much as about 22.8 cubic inches of pure water at its greatest density. The grain is one-five thousand seven hundred sixtieth of the troy pound. 7000 of these grains equal the pound avoirdupois.

TIME MEASURE.

The year of the calendar is the time required for the earth to pass from a point in its orbit, called the vernal equinox, to the same point again.

Since this point has a slight motion backward the earth makes a little less than a complete revolution about the sun each year. The astronomers call this the *tropical year;* its length is 365 d., 5 h., 48 min., 46.05 sec.

The word *day* has several meanings. The time required for the earth to turn once upon its axis is called a *sidereal*, or *star*, *day*. The time elapsing between two successive passages of the sun across the same meridian is called a *solar day;* it is a little longer than a sidereal day and is of variable length. The *average solar day* is the day referred to in the table of time measure. It is divided into 24 equal parts called hours.

The ancients were unable to determine the number of days in a year hence much confusion resulted. 46 B. C., Julius Caesar reformed the calendar, calling the year 365¼ days, the common year being 365 days, and every fourth year 366 days. The extra day was introduced by repeating the 24th of February. As this day was then the sixth before the first day of

March the years in which it was doubled were called *bissextile* years, or years with *two sixths*.

Since each year lacked about eleven minutes of being 365¼ days long, the addition of one day in four years was too much by about 45 minutes. Twenty-five such additions in a century would make an error of about three-fourths of a day. 1582 the error had amounted to about ten da

Pope Gregory corrected this error by ordering that the 5th of October of that year should be called the 15th. A hundred and seventy years passed before this reform was introduced into England and her colonies. The error had then amounted to eleven days, consequently dates before 1752 are occasionally called Old Style, or O. S.

The Gregorian calendar is not perfect, but the error is very slight as the extra day is added only to those years

(1.) Whose numbers are divisible by 4 and not by 400, and

(2.) Whose numbers are divisible by 400 and not by 4000.

The origin of the names of the months and of the days of the week can be found by consulting the dictionary. Curious pupils will want to know about them.

For measurement of angles and miscellaneous objects consult the usual text-books.

LONGITUDE AND TIME.

When the sun is on the meridian of any place, it is noon at that place.

(The equation of time is disregarded for obvious reasons). The time elapsing from one noon to the next is divided into twenty-four equal parts, each of which is called an hour. Clocks and watches are machines so constructed that they revolve indexes over a graduated circle called a dial, on which their motions are registered. Timepieces are usually regulated to show "sun time." When a watch marks twelve o'clock in the day we expect to find the sun on the meridian of the place at which the watch is set; hence one should be able to see by his watch how far the sun is from the meridian at any time.

Longitude is measured upon the circumference of the equator or of a parallel, east or west from an established meridian. These circles are perpendicular to the axis of the earth, and, since the earth revolves upon its axis, the circumference of each of these

circles revolves under the sun from noon to noon. Each circumference is divided into 360 equal parts called degrees, hence

An arc of fifteen degrees passes under the sun each hour, an arc of fifteen minutes each minute, and of fifteen seconds each second. We may then make the following table:

A difference of	15 degrees	in the	longitudes	of two places	makes a	difference of	1 hour	in their time.
"	15 minutes	"	"	"	"	1 minute	"	
"	15 seconds	"	"	"	"	1 second	"	

OR, STATING THE CONVERSE OF THIS TABLE WE HAVE:

A difference of	1 hour	in the	time	of two places	shows that they	have a	difference of	longitude of 15 degrees.
"	1 minute	"	"	"	"	"	"	15 min.
"	1 second	"	"	"	"	"	"	15 sec

The tables found in some arithmetics tell us that "15 degrees of lon. equal an hour of time, etc."

Such statements are obviously incorrect and misleading.

A is in lon. 50 degrees 24 min. east of Washington, and B is lon. 17 degrees 12 minutes west of Washington; what is their difference of lon., or, in other words, A is how far east of B, or B is how far west of A?

It seems strange that pupils should be bothered by so simple a problem, yet experience proves that they are. The difficulty arises, probably, from incorrect notions, or none at all, of what longitude is. If familiar units of measure, as the mile, the foot, etc., were used, there would be no trouble. The matter may be made perfectly clear by requiring pupils to find the distance between towns on opposite sides of them. If A and B are on the same side of the prime meridian, a similar illustration should be used.

The difference of longitude of A and B is 77 degrees and 36 minutes; what is the difference between the local times of the two places?

Since a difference of 15 degrees in the longitudes of two places makes a difference of one hour in their

times, a difference of 77 degrees makes a difference of as many hours as there are fifteens in seventy-seven; there are five fifteens in seventy-seven, with a remainder of two; hence, 5 hours and two degrees remaining; 2 degrees = 120 minutes; 120 minutes plus 36 minutes = 156 minutes, etc.

The difference in the times of two places is 2 hours 24 minutes; what is their difference of longitude?

Since a difference of one minute in the times of two places shows a difference of 15 minutes in their lon., a difference of any number of minutes of time shows a difference of 15 times as many minutes of lon. A difference of 24 minutes of time, therefore, shows a difference of 15 times 24 minutes of lon., or 360 minutes of lon., etc. (Since pupils are inclined to confuse the two kinds of minutes, the symbols should be employed instead of the words).

Since a difference of 15 degrees of lon. makes a difference of one hour in time, a difference of one degree in lon. makes a difference of 4 minutes in time, a difference of 1 minute in lon. makes a difference of 4 seconds in time, and a difference of 1 second in lon. makes a difference of four-fifteenths of a second in time.

These facts being mastered, work may often be shortened.

ILLUSTRATIONS.

1. The difference in lon. of two places being 18°, 25′, 30″, what is their difference in time?

Since a difference of 15 degrees of lon. makes a difference of an hour in time, a difference of 18 degrees of lon. makes a difference of an hour and 12 minutes in time. Since a difference of 15 minutes in lon. makes a difference of one minute in time, a difference of 25 minutes in lon. makes a difference of 1 minute and 40 seconds in time. 12 min.+1 min.=13 min. Since a difference of 15 seconds of lon. makes a difference of 1 second in time, a difference of 30 seconds of lon. makes a difference of 2 seconds in time; 40 sec.+2 sec.=42 sec.; hence, etc.

2. The difference in the times of two places is 6 hours, 26 min., 25 sec.; what is their difference in lon.?

(Fill out the following:)

A difference of 6 hours shows a difference of 90 degrees. A difference of 26 minutes shows a differ-

ence of 6 degrees, with a remainder of 2 minutes. A difference of 2 minutes of time shows a difference of 30 minutes of longitude, etc.

The matter should now be cleared up by numerous illustrations similar to the following:

Suppose that you should start from Chicago, your watch indicating Chicago time, and after traveling for a time you should find that the time at some station which you are passing is ten o'clock A. M. while your watch says nine o'clock A. M. Which way have you gone? How do you know? How far east or west have you traveled? How do you know?

Mariners employ these calculations in determining their longitude at sea. They are supplied with watches called chronometers, that run with remarkable accuracy. These watches indicate the time of the port of departure, or of some observatory, as Greenwich. Whenever a mariner looks at his chronometer, then, he sees Greenwich time. By means of his compass he can establish a north and south line. If the sky is clear, so that he can see the sun, he can tell when it is noon where he is as a farmer does—by noticing when his shadow makes a north

and south line. He knows that it is then noon. His chronometer tells him what time it is at his port of departure. He is thus enabled to determine his longitude. By similar observations, he can also ascertain his latitude. The inability to make these calculations, accounts for many disasters at sea—"The weather had been so cloudy that the captain had been unable for several days to take an observation."

CHAPTER XV.

PERCENTAGE.

Percentage is that part of Arithmetic which treats of computations by hundredths.

One per cent. of anything is one hundredth of it; two per cent. is two hundredths of it, etc.

Per cent. may be expressed in four ways:

1. As a decimal fraction.
2. As a common fraction.
3. By the use of the symbol of per cent.
4. By the words *per cent.*

Pupils should be required to express in these four ways any per cent., whether integral, fractional or mixed.

Percentage presents three general problems, or cases, and no more.

First General Problem: To find any per cent. of any number.

Second General Problem: To find what per cent. one number is of another.

Third General Problem: To find a number when some per cent. of it is given.

For each of these General Problems there are two solutions. Pupils should master both, and should let the character of the particular problem determine the one to be used. Each solution also involves certain subordinate problems that will appear in the discussion.

THE FIRST GENERAL PROBLEM.

First Method. Find one per cent. of the number and multiply the result by the number expressing the per cent.

The Subordinate Problem: To find one per cent. of a number. With integers this is done by moving the decimal point two places to the left; with fractions, by dividing by 100 in the most convenient method possible in the given case.

Second Method. Change the per cent. to a common

fraction in its lowest terms, and take such a part of the number as the fraction expresses.

Subordinate Problem: To change any per cent. to a common fraction. Erase the sign of per cent. and write 100 as a denominator.

If the number expressing the per cent. is fractional or mixed, this reduction will present a complex fraction. Pupils should become very expert in reducing these fractions to simple ones. A knowledge of the aliquot parts spoken of in an early article will be found very convenient in this connection.

ILLUSTRATIVE PROBLEMS.

1. What is 23 per cent. of 864?

One per cent. of 864 is 8.64, obtained by moving the decimal point two places to the left. 23 per cent. of 864 is 23 times 8.64, etc.

What is 13⅓ per cent. of 690?

13⅓ per cent. equals 13⅓ hundredths, or 40-300, or 2-15. 2-15 of 690=92.

THE SECOND GENERAL PROBLEM.

First Method. Find one per cent. of the second

number, and divide the first number by it. There are two subordinate problems in this solution:

(*a.*) Finding one per cent. of a number.

(*b.*) Dividing by a decimal fraction.

ILLUSTRATIVE PROBLEMS.

1. 18 is what per cent. of 72?

One per cent. of 72 is .72, obtained by moving the decimal point two places to the left. 18 is as many per cent. of 72 as 18 is times .72. 18=18.00. 18.00 is 25 times .72, hence 18 is 25 per cent. of 72.

It will be seen that the second subordinate problem is performed by reducing the dividend to the denomination of the divisor—a convenient plan in division of decimals.

This solution is very simple and can be performed rapidly. The pupils should perform a large number of problems, and the practice work should be continued until great facility is acquired. When the problem is given the first number should be written at the right of the second, with a curved line between them. One hundredth of the divisor should then be obtained, and the dividend should be reduced to hundredths. Insist that the decimal point shall

be in place. Do not permit pupils to say "I annex two zeroes." It should be clear to the pupil that *the dividend is reduced to hundredths.* When this work is completed the problem presents the following form: .72) 18.00 (The quotient is, of course, units.

If the numbers are fractional, the solution is the same. Thus: 3-5 is what per cent. of 7/8? One per cent. of 7/8 is 7-800. 3-5 is as many per cent. of 7/8 as 3-5 is times 7-800. Inverting the divisor, the problem assumes the form 3 5×800-7. Canceling the common factor five, the result is 480-7, or 68 4-7, hence 68 4-7 per cent.

Second Method. Find what part the first number is of the second and change this part to hundredths. It will be seen that there are two subordinate problems in this solution:

(*a*). Finding what part one number is of another.

(*b*). Changing a common fraction to hundredths.

These, like the preceding subordinate problems, were fully discussed before, consequently should not be new. If they are forgotten review them.

ILLUSTRATIVE PROBLEMS.

1. 7 is what per cent. of 15?

7 is 7-fifteenths of 15. 7-fifteenths=equals one-fifteenth of 7.00, or .46⅔ or 46⅔ per cent. The quotient in this case is hundredths.

If the numbers are fractional the solution is the same.

2. Four-fifteenths is what per cent. of seven-twelfths?

Four-fifteenths is sixteen-thirty-fifths of seven-twelfths. Change this fraction to hundredths as before.

The work in percentage is a constant review of common fractions. Little can be done unless that subject has been mastered.

The successful teacher sees far in advance of his class. He knows that the lesson of to-day is a preparation for the work of months hence, and so he builds against the time to come.

THE THIRD GENERAL PROBLEM.

As in the preceding, so in this there are two methods of solution.

ILLUSTRATIVE PROBLEMS.

1. 35 is 7 per cent. of what number?

Since 35 is 7 per cent. of some number, one per

cent. of that number is one-seventh of 35, or 5. 100 per cent. of the required number is 100 times 5, or 500.

2. 36 is 16 per cent. of what number?

16 per cent. = sixteen hundredths, or four twenty-fifths. Since 36 is four twenty fifths of some number, one twenty-fifth of that number is one-fourth of 36, etc.

Let the pupils state the two methods of solution of this General Problem.

Percentage presents no other problems than these. When a problem is presented the pupil must determine where it belongs. If he can locate it the solution will not be difficult.

$4.80 is 33⅓ per cent. more than what number?

Problems like the above are often assigned to a Fourth case; they are simply forms of the third General Problem. A fuller statement of the problem is:

$4.80 is 33⅓ per cent. of some number more than that number; what is the number?

If $4.80 is 33⅓ per cent. of some number more than the number, it is 100 per cent. of the numbet + 33⅓ per cent. of the number, or 133⅓ per cent. of the number.

28 is 20 per cent. less than what? means 28 is 20 per cent. of some number less than the number; what is the number?

If 28 is 20 per cent of some number less than the number, it is 100 per cent. of the number—20 per cent of the number, or, it is 80 per cent. of the number.

Insist that the pupils shall not use the term per cent. without telling "per cent. of what."

CHAPTER XVI.

LOSS AND GAIN.

The problems in percentage that give most trouble are those in Loss and Gain; but if the pupil can always answer the question "per cent. of what?" not much difficulty should be encountered. Every problem falls under one of the general problems given.

Permit no such expressions as "let 100 per cent. = the number."

Such solutions are algebraic and should not be tolerated in arithmetic.

In reading the problem the pupil should be required frequently to supplement the test by expressing per cent. of what. Require both forms of solution

but use the one that is most convenient for the particular problem under consideration.

The most troublesome problems are of the second kind, consequently more of these should be wrought than of the others.

ANALYSIS.

1. For what must I sell a horse that cost me $150 to gain 35 per cent.?

This is a problem of the first kind. I must sell the horse for $150+35 per cent. of $150. One per cent. of $150 is, etc.

Or, 35 per cent. = 7-20. I must sell the horse for $150+7-20 of $150, etc.

2. Bought sugar at 8 cents a pound; the wastage is 12½ per cent.; how must I sell it so as to gain 25 per cent.?

This is a problem of the first kind. The wastage is 12½ per cent., or ⅛, of each pound purchased; at how much per pound must I sell it to gain 25 per cent., or ¼, of the cost of each pound?

Since ⅛ of each pound is lost, only ⅞ of each pound is left, hence ⅞ of a pound costs 8 cents. ⅛ of a pound, then, costs 1-7 of 8 cents, or 8-7 cents,

and 8-8 of a pound costs 64-7 cents, or 9 1-7 cents. It must be sold for 9 1-7 cents + ¼ of 9 1-7, etc.

3. I buy at $7 and sell at $8.50; what is the per cent. of gain?

This is a problem of the second kind. If I buy at $7 and sell at $8.50 I gain $1.50. $1.50 is what per cent. of $7? Solve by both methods.

4. I buy a number of pounds of sugar. The wastage is 20 per cent., and it is sold at 40 per cent. above cost; what is the gain per cent.?

This is a problem of the second kind. If 20 per cent., or 1-5, of the sugar is lost, the remaining 4-5 cost as much as the whole; then each pound remaining cost 5-4 of the original price per pound. If each remaining pound is sold for 140 per cent., or 7-5, of the original price per pound the gain on each pound is 7-5—5-4, or 3-20, of the original price. 3-20 of the original price is what per cent. of 5-4 of the original price?

5. I bought goods for $7.29. At what price must must I mark them that I may fall 10 per cent., lose 10 per cent. in bad debts, and yet gain 10 per cent.?

Problems of this kind are not unusual. The only difficulty is "per cent. of what?" A full statement is as follows:

I bought goods at $7.29; at what price must I mark them that I may gain 10 per cent. of the cost price, lose 10 per cent. of the selling price, and fall 10 per cent. of the marked price?

If the gain is 10 per cent. of the cost, it is $.729; hence $7.29+$ 729=$8.019 is collected. Which kind of problem is this part?

If 10 per cent., or 1-10, of the selling price is not collected, then $8.019 is 9 10 of the selling price. This part is a problem of the third kind.

If $8.019 is 9-10 of the selling price, 1-10 of the selling price is 1·9 of $8.019, ar $.891; 10-10 of the selling price is 10 times $.891, or $8.91.

If I fall 1-10 of the marked price, $.91 must be 9-10 of the marked price, etc.

The straight-line analysis should be employed where it is possible to use it. Go through the foregoing analysis indicating the operation. When it is completed the problem will have assumed the following form:

$$\frac{\$7.29 \times 11 \times 10 \times 10}{10 \times 9 \times 9}$$

Employing cancellation the work is materially shortened.

CHAPTER XVII.

COMMISSION.

Each of the applications of percentage furnishes problems of the three kinds.

Those in commission and stock investments seem to give most trouble.

Commission problems may be characterized as "selling problems" and "buying problems," the second being the more difficult.

The old question "Per cent. of what?" again presents itself for an answer. This understood the difficulties vanish.

A commission merchant sells for a customer and is to receive two per cent. The meaning simply is that he is to retain two per cent. of all moneys that he receives, as compensation for his services. Few

pupils have trouble with this problem. The solution is entirely simple when the problem is recognized. The "buying problems," however, perplex the dull pupils, and are an occasional snare to the bright ones.

Money is sent to an agent to be invested with the understanding that the amount sent covers the investment and the agent's commission. The books are usually silent upon the latter point, leaving it to be inferred by the pupil.

A man sends a commission merchant a sum of money which he is to invest at 3 per cent. What is the agent's commission and how much did he invest?

The statement is a little obscure, because the language used is technical.

There are two methods of solution.

(*a*) For each dollar which the agent invests he charges 3 cents commission; hence each dollar invested in the business venture uses up one dollar and three cents of the money sent. As many dollars will be invested as the money sent is times one dollar and three cents, etc. The commission is, of course, the difference between the money sent and the money invested.

(*b*) The commission is three per cent. of the money invested. The money sent, then, is 103 per cent. of the amount to be invested. One per cent of the amount to be invested is one one-hundred-third of the amount sent, and the investment is 100—103 of the same amount. The commission is 3—103 of the amount sent, or 3—100 of the investment.

By this method the commission may be obtained without finding the investment.

An exceedingly useful exercise consists in the classification of problems. Let a pupil read a problem and "locate it," without stopping to solve it. The examples in several different arithmetics may be examined in this manner and readiness acquired by the increased familiarity.

The problems in commission may be illustrated in a very simple manner. Let two pupils act as dealers and a third as a commission merchant or "middleman." Call them respectively A, B and C. Pieces of paper suitably prepared may represent currency. A sends goods to C to be sold at a commission of two per cent. B buys them.

From every dollar that C receives from B, he takes two cents and sends the remainder to A; hence A

receives but ninety-eight per cent. of what his goods sell for in the market.

If C were collecting money for A the operation would be the same. This form of the problem is very simple, and is easily mastered.

Suppose, however, that A sends money to C to invest for him at a commission of two per cent., with the understanding that the money sent includes the investment and the commission upon it. The pupils are inclined to think that the commission should be reckoned upon the whole amount sent, and subtracted from it, and that only the remainder should be invested.

They do not remember that the agent is to get two cents for investing one dollar, not for investing ninety-eight cents.

Pupil C takes one dollar from the money that A has sent him and buys something from B. To pay himself for his work he takes two cents from the remaining money and puts it into his pocket. He has now invested one dollar for A, but has used up $1.02 of his money; so, for every dollar that he invests he uses up $1.02, hence he can invest as many dollars as the money sent is times $1.02.

For the other form of solution see *b*, p. 141.

The following analysis of problems found in Ray's Higher Arithmetic will illustrate the methods:

1. Bought flour for A; my whole bill was $5802.57, including charges $76.85 and commission $148.72; find the rate of commission.

The whole bill must include the investment, the commission upon it, and the remaining charges. The commission and charges amount to $225.75. The investment, then, is $5802.57—$225.57, or $5577 The question now is, $148.72 is what per cent. of $5577? Under what general problem does this fall? Work by either method.

2. An agent sold my corn, and, after reserving his commission, which was 3 per cent. for buying and 3 per cent. for selling, he invested the remainder in corn; his charge was $12; for what was the first corn sold?

Since he received a commission of 3 per cent. of what the corn sold for, only 97 per cent. of this amount remained. Since he is to receive 3 per cent. of what the second corn costs, 97 per cent. of what the first corn brought is 103 per cent. of what the second cost. One per cent. of what the second cost is 1-103 of 97 per cent. of what the first brought; 100 per cent. of what the second cost is 100-103 of

97 per cent. of what the first brought. The commission on the second, then, is 3·103 of 97 per cent. of what the first brought, or 291-103 per cent., or 2 85-103 per cent. of what the first brought. The commission, then, is 3 per cent. of the cash of the first plus 2 85-103 per cent. of the same, or 5 85-103 per cent. $12, then, is 5 85-103 per cent. of the cost of the first corn. 5 85-103 per cent. = 600-103 per cent. 1-103 per cent. of what the first cost = 1-600 of $12, and 100 per cent. of the cost of the first is 100×103 600 of $12, or, omitting common factors, $206.

CHAPTER XVIII.

STOCK INVESTMENTS.

This is another of the topics that trouble pupils. The same set of problems recurs. All, however, are assignable to one or another of the Three General Problems. The trouble here as elsewere is to answer the old question "Per cent. of what?"

The meaning of all terms used must be entirely comprehended.

Since there are five terms in common use we may expect five problems.

These terms are Investment, Market Value, Rate of Interest, Rate of Income, Income.

Most of these problems are quite simple. Those are most difficult that fall under the Second General Problem. The following will illustrate.

What is the rate of income on 6 per cent. bonds bought at 112?

These bonds pay as interest 6 per cent. of their par value. They are bought at 112 per cent. of their par value. The question is 6 per cent. of a number is what per cent. of 112 per cent. of the same number?

If pupils do not grasp the subject take a special case.

I buy a hundred dollar six per cent. bond at 112 per cent. of its par value. What rate of interest do I receive on my investment?

The bond cost $112. It pays $6 a year as interest. $6 is what per cent. of $112?

Problems like the following are sometimes troublesome:

At what rate must 4 per cent. bonds be bought to make the investment pay 10 per cent.?

The question is, 4 per cent. of anything is 10 per cent. of what per cent. of it? or 4 is 10 per cent. of what?

If 4 per cent. of the par value is 10 per cent. of the investment 1 per cent. of the investment is 1-10 of 4 per cent. of the par value, or 4-10 per cent. of the par value, etc.

This may also be made simpler by making a speciai problem.

At what per cent. of the par value must I buy a 4 per cent. $100 bond that I may receive 10 per cent. interest on my investment?

This bond pays $4 a year as interest. $4, then, is 10 per cent. of the price to be paid for it. 1 per cent. of the price to be paid for it is 1-10 of $4 or 40 cents. 100 per cent. of the price to be paid for it is 100 times 40 cents, or $40. $40 dollars is 40 per cent. of $100, hence I must buy the bond for 40 per cent. of the par value, or 60 per cent. discount.

The following problem from Ray's Higher Arith. illustrates the necessity of knowing "per cent. of what."

Suppose 10 per cent. State Stock 20 per cent. better in market than 4 per cent. railroad stock; if A's income be $500 from each, how much money has he paid for each, the whole investment bringing 6 2-333 per cent.?

If his annual income from the state stock is $500, then $500 must be 10 per cent. of the par value of his state stock. If $500 is 1-10 of the par value then the par value is $5000.

Reasoning similarly he must have $12,500 of the railroad stock.

If the market value of the state stock is 20 per cent. of the market value of the railroad stock more than the market value of the railroad stock it is 6-5 of it. If the market price of the two were the same the state stock would cost 2-5 as much as the railroad stock, since these are 2-5 as much of it. But since it cost 6-5 as much in market, his state stock must have cost 6-5 of 2-5, or, 12-25 as much as the railroad stock.

The two, then, must have cost 25-25+12-25 or 37-25 as much as the railroad stock.

His whole income is $1000. This is 6 2-333 per cent. of his investment. 6 2-333 per cent. = 2000-333 per cent. If $1000 is 2000-333. per cent of his investment 1-333 per cent. of his investment is 1-2000 of $1000, or $½. 333-333 per cent. or one per cent. of the investment is 333 times $½, or $166.50, and 100 per cent. of the investment is 100 times $166.50 or $16650. But the whole investment is 37-25 of the cost of the railroad stock, hence 1-25 of the cost of the railroad stock is 1-37 of $16650=$450 and 25-25= 25 times $450, or $11250.

What is the cost of the other?

This may be shortened by indicating the operations as the analysis progresses, and omitting common factors at the close.

$$\frac{\$1000 \times 333 \times 100 \times 25}{2000 \times 37}$$

CHAPTER XIX.

INTEREST.

The computation of interest affords one of the most common applications of the problems of percentage. The pupils should understand thoroughly what is meant by "ten per cent. per annum." It means simply that for the use of a given thing, whatever it may be, for one year, ten per cent. of that thing is to be allowed. For fractional parts of a year, then, fractional parts of ten per cent. of the given thing are to be paid, etc.; hence these problems are of the first kind.

Our text-books present several methods of computing interest. One, made perfectly familiar, is sufficient, and is immeasurably better than a super-

ficial knowledge of three or four. "A ten per cent. method" has found its way into few or none of the books. Those unfamiliar with it will judge of its worth after examining the following statements:

1. The interest for one year is found by moving the decimal point one place to the left.

2. Since three hundred and sixty days are considered a year, in computing interest ordinarily, it follows that the interest for thirty six days is one hundredth of the principal, and is found by moving the decimal point two places to the left.

3. The pupil, then, must always keep in mind these two periods—one year, and thirty-six days. For any number of years multiply one-tenth of the principal by the given number; for any number of months take parts of the interest for one year; for any number of days take parts of the interest for thirty-six days.

For rapid and accurate work "a method" is necessary and should be adhered to rigidly. The following illustrative problems will indicate the methods in all cases:

$2.6.5.80	1 year.	8 mos.	28 days.
2 6.5 80			
8.8 6		—	—
8.8 6		4	12
.8 86		4	12
.8 86			4
.2 95			

$46.37

The time is more than one year and less than two. In such cases:

First. Move the decimal point one place to the left.

Second. Draw a horizontal line below the number of months, and separate this number into convenient divisors of twelve.

Third. Take such parts of the interest for one year as the divisors are of twelve, and write the results for addition to the interest for one year.

Fourth. Draw a horizontal line below the number of days and separate the number into convenient divisors of thirty-six.

Fifth. Indicate one-tenth of the interest for one year and take such parts of this result as the divisors are of thirty-six.

Use no unnecessary figures. Observe the model carefully.

2. If the time is less than one year:

First. Find one-tenth of the principal and draw a horizontal line below it.

Second. Separate months and days as before, obtain results in same way and write them below the line for addition.

$4.8.6.50	10 months.	14 days.
2 4.3 25	6	12
1 6.2 16	4	2
1.6 21		
.2 70		
$42.43		

3. If the time is two years or more:

First. Find one-tenth of principal and draw a horizontal line below it.

Second. Below this line write as many times the interest for one year as there are years.

Third. For months and days proceed as before.

$5.2.6.70	4 years.	7 months.	17 days.
21 0.6 8		6	12
2 6.3 35		1	4
4.3 89			1
1.7 55			
.5 85			
.1 46			
$243.89			

Observe that in the above final results no mills are written. The sum of the mills is found and in reducing it to cents, a remainder of five or more is called one cent. A smaller remainder is disregarded.

If, in the final result, the decimal point be removed one place to the left, the interest for the given time at one per cent. is obtained. This multiplied by the given rate will give the result required.

The only objection to the ten per cent. method as a general method is that in problems involving several calculations, as in partial payments, pupils forget, sometimes, to find the interest at the required rate after finding it at 10 per cent.

A GENERAL PROBLEM.

Find the interest on the principal for one year at the given rate. If the rate is 8 per cent., take 8 hundredths of the principal; a similar process is employed for any other rate. In all other respects the operations are the same as in the ten per cent. method. To find the interest for any number of years multiply the interest for one year by the given number of years. Separate the months into convenient divisors of twelve and take such a part of the interest for a year as the number of months is of twelve. Take one tenth of the interest for a year and the interest for thirty-six days is obtained. Separate the number of days into aliquot parts of 36 and proceed as before.

SIX-PER-CENT. METHOD.

Since any principal at 6 per cent. gains 6 hundredths of itself in 12 months it will gain one hundredth of itself in 2 months and will double itself in 200 months. It will, therefore, gain one thousandths of itself in 6 days, or one tenth of two months. The period of time to be kept in mind with especial care are 200 months, 2 months and 6 days, the principal

doubling itself, gaining one hundredth and one thousandth of itself in the respective periods.

FORM OF WORK.

Reduce the years to months and to this result add the given months. Draw a line below the number and separate it into divisors of 200. Separate the days, in the same manner, into divisors of 60 days.

Problem. What is the interest on $5680 for 5 yrs. 5 months and 22 days?

$5680	5 years, 5 months,= 65 months.	22 d.
1420.	50	20
284.	10	2
142.	5	
18.933		
1.893		
$1866.83		

An explanation similar to the following will be found useful:

5 yrs. and 5 mos. equal 65 mos., which I separate into 50 mos., 10 mos. and 5 mos. In 50 mos. the principal will gain one-fourth of itself; in 10 mos. it

will gain one-fifth of this amount; in 5 mos. it will gain one-half of the interest for 10 mos. I separate 22 days into 20 days and two days. Since 20 days is ⅓ of 60 days, the interest is ⅓ of 1-100 of the principal; the interest for 2 days is 1-10 of the interest for 20 days. Combining these several results, etc.

Facility cannot be expected until questions like the following can be answered with perfect readiness.

The principal gains what part of itself in 100 mos.? 20 mos.? 66⅔ mos.? 33⅓ mos.? 20 days? 3 days?

TO FIND THE TIME BETWEEN TWO DATES.

Find the full number of years, the full number of months, and count the remaining days.

The following illustrations will show the forms which the problem may assume:

1. What is the time between Jan. 5, 1874, and April 16, 1878?

METHOD.

From Jan. 5, '74, to Jan. 5, '78, is 4 years.

From Jan. 5, '78, to Apr. 5, '78, is 3 months.

From Apr. 5, '78, to Apr. 16, '78, is 11 days.

2. What is the time between March 15, '72, and Feb. 20, '76

METHOD.

From March 15, '72, to March 15, '75, is 3 years.

From March 15, '75, to Feb. 15, '76, is 11 mos.

From Feb. 15, '76, to Feb. 20, '76, is 5 days.

3 What is the time between Sept. 20, '68, and Aug. 10, '73?

METHOD.

From Sept. 20, '68, to Sept. 20, '72, is 4 years.

From Sept. 20, '72, to July 20, '73, is 10 months.

From July 20, '73, to Aug. 10, '73, is 21 days.

PARTIAL PAYMENTS.

Pupils should be so instructed that they shall be able to prepare notes of any form promptly, neatly and accurately. Punctuation, legibility, form of paper, and all the details should receive special attention. Notes should be written with ink and handed to the teacher for criticism.

The reasonableness of the U. S. Rule should be apparent.

Much practice in performing problems in partial payments is needed before facility and accuracy can be acquired.

Some definite form should be selected and insisted upon. The following is suggested:

$725.50. *Bloomington, Ill., Feb.* 15, 1880.

Two years after date, for value received, I promise to pay Edwin C. Hewett, or order, Seven Hundred

Twenty-Five and 50-100 Dollars, with interest at eight per cent. per annum from date.

JOHN SMITH.

Indorsements:

July 20, 1880,	$50.00
Sept. 12, 1880,	62.50
Jan. 6, 1880,	42.00
Dec. 24, 1881,	6.00
Feb. 1, 1882,	100.00

What was due Feb. 15, 1882?

With this problem before the pupils, have them put it upon the board, slate or paper, in the following form:

$725.50 at 8 per cent.

		m.	d.
	Feb. 15, 1880		
$ 50.00....	July 20, 1880	5.	5
$ 62.50....	Sept. 12, 1880	1.	23
$ 42.00....	Jan. 6, 1881	3.	25
$ 6.00....	Dec. 24, 1881	11.	18
$100.00....	Feb. 1, 1882	1.	8
	Feb. 15, 1882		14

In solving the problem use the form given in the ordinary text books. No multiplication should be permitted except in obtaining the interest for a year. Reject all results below tenths of mills.

CHAPTER XX.

PRESENT WORTH.

The present worth of a sum of money due at a future time, *and not bearing interest*, is that sum which put at interest at the given rate will amount to the obligation in the given time.

There are two methods of solution.

1. What is the present worth of a debt of two hundred dollars, due in two years and six months, and not bearing interest, money being worth 8 per cent. per annum?

(*a*) If one dollar be put at interest at 8 per cent. for two years and 6 months it will amount to $1.20. One dollar, then, is the present worth of $1.20, due in two years and six months. The present worth of $200, then, due at the same time must be as many

dollars as there are times $1.20 in $200. There are 166.66 times $1.20 in $200; hence the pressent worth of $200 under the above conditions is $166.66.

The justice of the above is obvious. If I owe a debt which is not due for two and a half years and I am not to pay interest upon it, I am entitled to the use of the amount for that time. If I pay the debt in full before its maturity I lose the use of the money for the given time. If, however, I pay the creditor a sum which he can put at interest at the given rate for the given time and which will amount to the debt at its maturity neither loses.

The difference between the present worth and the amount of the debt is the Discount.

(*b*) A second method of solution.

Any sum of money put at interest at 8 per cent. for two and a half years will amount to six-fifths of itself. The amount, then, is six-fifths of its present worth. One-fifth of the present worth is one-sixth of the amount. Five fifths of the present worth equals five-sixths of the amount: hence, in this case, the present worth is five-sixths of $200.

As will be seen, the example falls under the third General Problem of Percentage.

This method enables pupils to obtain the discount

without finding the present worth. The present worth is five-sixths of the amount. The discount, then, is one-sixth of the amount.

Suppose the debt is bearing interest. If so, find the amount which is to be paid at the maturity of the note and proceed as above.

True Discount is one of the difficult applications of percentage.

CHAPTER XXI.

BANK DISCOUNT.

The chief cause of difficulty in this and kindred topics is the ignorance of pupils in regard to the practical details of business matters. Although ninety-five per cent. of the business of the country is carried on by means of checks and drafts, yet few pupils have seen either. Teachers should instruct the pupils in such matters in order that the work of the class-room may have that air of reality without which little good is reached.

Many teachers are as ignorant as their pupils. They must inform themselves if they expect to succeed in making these matters clear. Obtain accurate and clear answers to the following questions:

1. What is a bank?

2. How many kinds of banks are there?

3. By what other name are banks of issue designated?

4. Suppose a bank of issue should fail, would the holders of its notes lose anything? Why?

5. Of what advantage is a bank of deposit?

6. In how many ways may money be deposited in a bank?

7. What is a certificate of deposit? Write one.

8. If one has deposited money and has taken a certificate, how can he get the money?

9. Can anyone else get it? If so, how?

10. What is a "pass book?" What use is made of it.

11. What is a check? Write one.

Banks can, of course, derive no profit from deposits unless they can use them. Experience demonstrates that they may safely lend a certain part of their deposits and yet be able to meet all ordinary demands upon them; hence lending money is the chief business of a bank. Bankers conduct this business somewhat differently from private individuals. The borrower gives a note for a certain time, without interest until due. The banker calculates the interest

upon the face of the note for the given time and for three days additional, called "days of grace," subtracts this interest from the face, and gives the borrower the remainder.

This is called discounting notes.

BANK DISCOUNT presents two problems, the second of which is the more difficult. There is no reason, except custom, why the first should differ from the True Discount. It results in an advantage to the banker, and is a trifle easier to compute.

(*a*) To find the present worth of a note discounted at a bank, and not bearing interest, find the simple interest on the face of the note for the given time plus three days, and subtract it from it. The banker thus receives a higher rate than the one specified. For example:

Suppose the face of the note to be $200, the time 60 days, and the rate 6 per cent. The interest on $200 for 63 days is $2.10. The avails, then, are $197.90. The banker pays that amount for the note. At its maturity the debtor gives $200 for the same note, thus paying $2.10 interest on $197.90 for 63 days; the rate, then, is something more than 6 per cent.

It is a common saying that the interest in such

cases is paid in advance. This is obviously incorrect since the debtor pays nothing until the maturity of the note. He simply pays a rate of interest a little higher than the nominal rate.

The second problem, as has been stated, is more confusing.

(*b*) To find the face of a note when the avails are given.

For what sum must a note be given that will yield $260 when discounted at 6 per cent. for 90 days?

The simple interest on one dollar for 93 days at 6 per cent. is $.0155. A note whose face is one dollar, consequently, will yield $.9845. To yield $260 the face of the note must be as many times one dollar, as $260 is times $.9845, etc.

This problem may also be performed by the second method employed in True Discount.

Problems like the following are very helpful in teaching pupils the philosophy of this kind of work The old question "Per cent. of what?" is the one that is constantly presented.

1. What is the rate of interest per annum on a 30 day note when discounted at 1 per cent. a month?

The discount is .011 of the face of the note. The

avails are .989 of the face of the note. The interest is 11-989 of the avails. The time, however, is 33 days. For 3 days it would be 1-11 of 11-989 of the avails; for 30 days it would be 10 times this amount, and for 360 days, 12 times this amount.

The work may be shown in the following form:

$\frac{11\times10\times12}{989\times11}$ The resulting fraction is $\frac{120}{989}$ which changed to per cent. equals—what?

This, like some of the problems previously noticed, may be made clearer by a special case.

Suppose the note were for $100. The discount, then, would be $1.10. The avails would be $98.90. The person selling the note would receive $98.90 for it. When the note is paid, the person discounting it would receive $100 for it; that is, $1.10 more than he paid for it. This $1.10, consequently, is the interest on $98.90 for 33 days. At the same rate the interest for three days would be 1-11 of $1.10, or $.10 For 360 days it would be 120 times $.10 or $12.

If $12 be paid for the use of $98.90 for one year what is the rate?

(2) What rate of discount on a 60 day note will yield 6 per cent. per annum?

Since the interest is 6 per cent. of the avails for 360 days, it is twenty-one twentieths of one per cent. of the avails for 63 days. The face of the note is 101.05 per cent. of the avails and the avails 10,000-10,105 of the face of the note. The discount is 105-10,105 of the face of the note. This being the discount for 63 days it is 120-21 of this for 360 days.

Reduce and change to per cent.

SPECIAL PROBLEM.

Suppose the above note were for $100. What must be the rate of discount to make it yield interest at the rate of 6 per cent. on the avails?

$100 is the sum of the avails of the note and 6 per cent. interest on the avails for 63 days.

Since any principal at 6 per cent. amounts, in 63 days, to 101.05 per cent., or 10105-10000 of itself, $100 must be 10,105-10,000 of the avails. 1-10000 of the avails is 1-10105 of $100, and 10000-10000 of the avails must be 10000-10105 of $100. The discount then is 105-10105 of $100. Since this is the discount for 63 days, for 3 days it is 1-21 of 105-10105 of a $100, or canceling 1-201, of $100. For 360 days it is 120 2021 of $100. 120-2021 = what per cent.?

Another kind of business transacted by banks is the collection of debts.

A in one place has a bill against B in another. A gives his bank an order upon B. The bank sends this order to a bank in the town in which B resides and it is presented for payment. Since B is anxious to maintain a reputation for prompt payment he is more likely to pay the bill than if it were sent directly to him by A. In such a case A is said to "draw upon" B. If B should refuse to pay the draft the collector would go to a notary and make an affidavit to that effect. The affidavit would be attached to the note and returned to the bank sending it. The draft is then said to be "protested." Business men are naturally reluctant to have their bills go to "protest" unless there is a good reason for it.

A third kind of business in which banks engage is the sale and purchase of Bills of Exchange.

CHAPTER XXII.

EXCHANGE.

Suppose that A in Bloomington, desires to pay B, in New York, a sum of money; he may do this in any one of several ways.

1. He may send the money by express.
2. He may send postal money orders.
3. He may buy a bill of exchange.

Let us see how a system of exchange may grow up between two cities.

Suppose that A can find some one in Bloomington to whom some one in New York owes an amount equal to the amount that A owes B. A can purchase of him an order on this New York creditor and send it to B. B can receive his pay on presentation of this order. Both debts are thus discharged without the double transmission of money. The convenience of this method is apparent.

Since it is not possible always to find persons in the relations supposed above, a banker in Bloomington and another in New York arrange to pay orders that each may draw upon the other.

All that is necessary now is for A to purchase of his bank an order upon its New York "Correspondent" and sent it to B, who presents it and receives his pay.

The New York bank in turn may sell orders upon the Bloomington bank.

This is the essence of all systems of exchange, whether domestic or foreign.

These orders are called Drafts, or Bills of Exchange.

If the Bloomington bank sells more drafts upon the New York bank than it pays for it, money must be sent to balance the account.

What is the difference between a check and a draft? What is the difference between a domestic draft and a foreign draft?

Drafts are of two kinds—Sight and Time. A Sight Draft is payable upon presentation.

A Time Draft is payable a specified time after "sight," that is, after the draft has been presented and the proper officer has written across the face the

word *accepted*, the date of acceptance, and the signature of the banking company.

A time draft should be cheaper than a sight draft for the same amount, since the seller has the use of the purchaser's money for a specified time before the draft is redeemed.

When will sight drafts be at a premium? When at a discount?

Exchange presents the same problems as Bank Discount.

1. What is the cost of $3600 sight exchange on New York at 1-10 per cent. premium?

The draft will cost $3600 plus 1-10 per cent. of $3600.

1 per cent. of $3600 is $36. 1-10 per cent. of $3600 is 1-10 of $36, or $3.60. The draft will cost $3603.60.

If exchange were at a discount instead of a premium the method would be the same.

2. How large a sight draft on New York will $3600 buy, exchange being 1-10 per cent. premium?

(*a*) A draft for one dollar will cost one dollar plus 1-10 per cent. of a dollar, or $1.001. $3600 will buy a draft whose face is as many times one dollar as $3600 is times $1.001.

(*b*) $3600 is the face of the draft plus 1-10 per cent. of the face. It is, then, 100.1 per cent., or 1001-1000 of the face of the draft.

1-1000 of the face of the draft is 1-1001 of $3600. 1000-1000 of the face of the draft is 1000-1001 of $3600.

The same problems occur with time drafts, with the added element of bank discount.

1. What will a 60-day draft for $6000 cost at ¾ per cent. premium, interest at 6 per cent.?

(*a*) Since this draft is not payable for 63 days, the purchaser should be allowed interest on his money for that length of time. It is customary to discount the draft for the time it has to run. If the exchange were neither at discount nor premium the purchaser would pay for the draft $6000 minus the interest on $6000 for 63 days at 6 per cent., or $5937. Since exchange is at ¾ per cent. premium, he must pay $5937 plus ¾ per cent. of $6000, or $5982.

(*b*) A draft for one dollar will cost one dollar plus ¾ per cent. of a dollar minus the bank discount of a dollar for 63 days at 6 per cent. A draft for $6000 will cost 6000 times this amount.

2. What is the face of a 90 day draft that may be

purchased for $4320.50, exchange ⅛ per cent. discount, interest 6 per cent.?

(*a*) A draft whose face is one dollar will cost one dollar minus ⅛ per cent. of a dollar, minus the bank discount of one dollar for 93 days; this equals $.98325; $4320.50 will buy a draft whose face is as many dollars as $4320.50 is times $.98325, which is, etc.

(*b*) The bank discount of any principal for 93 days at 6 per cent. is .0155 of that principal. The exchange discount is ⅛ per cent., or .00125 of the principal. Their sum is .01675,—the entire discount. The draft will cost its face minus .01675 of its face, or .98325 of its face; but the draft is to cost $4320.50. $4320.50, then, is .98325 of the par value of the draft. .00001 of the par value is 1-98325 of $4320.50, and the par value is 100,000 times this amount.

(3) A 30-day draft for $4000 costs $4010, interest 6 per cent.: what is the rate of exchange?

The bank discount of $4000 for 33 days is $22.

The premium is the sum of $22 and $10, or $32. $32 is what per cent. of $4000?

(4) A 60-day draft for $5650 costs $5647.175; exchange being one per cent. premium, what is the rate of interest?

The premium is $56.50. The interest, then, is $56.50 plus the difference between $5650 and $5647.175, which is $2.825. $56.50+$2.825=$59.325.

This sum is the interest on $5650 for 63 days. The interest for 3 days would be 1·21 of this amount, and for 360 days would be 120 times that result. The work may be expressed as follows:

$$\frac{\$59.325\times 120}{21}$$

Cancelling and multiplying, the result is $339.

The question now is, $339 is what per cent. of $5650?

What is it?

ARBITRATION OF EXCHANGE.

Define Circular Exchange. What is its advantage?

Define the Arbitration of Exchange.

The following problems from Ray's Higher Arithmetic will illustrate the subject:

1. A Louisville merchant has $10,000 due him in Charleston. Exchange on Charleston is ¼ per cent. premium. Instead of drawing directly, he advises his debtor to remit to his agent in New York at ⅜ per cent. premium, on whom he immediately draws

at 12 da., and sells the bill at 1½ per cent. premium, interest off at 6 per cent. What does he realize in this way, and what gain over the direct exchange?

The Louisville merchant has $10,000 to his credit in Charleston. He may draw a draft on his debtor and sell it at ¼ per cent. premium, receiving for it $10,025.

He advises his debtor, instead, to invest the $10,000 in New York exchange which, in Charleston, is at ⅜ per cent. premium. The $10,000 will buy a sight draft on New York for $9962.64. (How found?) This draft is sent to the New York agent of the Louisville merchant. The credit is now transferred from Charleston to New York, where the merchant has a credit of $9962.64. He now draws a 12 day draft on his agent for that amount, and at once sells it at 1½ per cent. premium less the bank discount for 15 days at 6 per cent. interest.

The bank discount is ¼ per cent. of the face of the draft.

The net premium, then, is 1¼ per cent.; he consequently receives for his draft 101¼ per cent. of $9962.64. What is the gain over the first method?

2. A merchant of St. Louis owes $7165.80 to a

Baltimore merchant, the amount being due in St. Louis. Exchange on Baltimore is ¼ per cent. premium; but, on New Orleans it is ⅛ per cent. premium; from New Orleans to Havana ⅛ per cent. discount; from Havana to Baltimore ¼ per cent. discount. What will be the value in Baltimore by each method, and how much better is the circular?

If the St. Louis merchant invests the $7165.80 in Baltimore exchange he will purchase a draft whose face is $7147.93, (How obtained?) and send it to his creditor.

He is directed to buy a draft on New Orleans, ⅛ per cent. premium.

With $7165.80 he can buy a draft whose face is $7156.85.

This draft is sent to a bank in New Orleans with directions to issue a draft on Havana for the amount that it will purchase.

Havana exchange being ⅛ per cent. discount $7156.85 will buy a draft whose face is $7165.81. This draft is sent to a Havana bank with directions to invest its proceeds in Baltimore exchange, which is ¼ per cent. discount. $7165.81 will buy a Baltimore draft whose face is $7183.77. This draft

is sent to the Baltimore creditor, who presents it and receives payment.

What is his gain by the circular method?

The following method is shorter, as it employs cancellation.

Since exchange on New Orleans is ⅛ per cent. premium, its cost is 100⅛ per cent., or 801-100 of its par value. The face of the draft, then, is 800-801 of its cost. $7165.80 will buy a draft on New Orleans whose face is 800-801 of $7165.80.

Since exchange on Havana is, in New Orleans, at ⅛ per cent. discount, its cost is 99⅞ per cent., or 799-800 of its face. Its face, consequently, is 800-799 of its cost, hence the New Orleans draft will purchase a draft on Havana for 800-799 of 801-800 of $7165.80.

Since Baltimore exchange is, in Havana, at ¼ per cent. discount, its cost is 99¾ per cent., or 399-400 of its face, hence its face is 400-399 of its cost.

The Havana draft, therefore, will purchase a Baltimore draft whose face is 400-399 of 800-799 of 801-800 of $7165.80.

This expression reduced will give the face of the Baltimore draft that can be obtained by the circular method.

CHAPTER XXIII.

EQUATION OF PAYMENTS.

The Equation of Payments is the process of finding a time at which several sums due at different times, and not bearing interest, can be paid, without loss to debtor or creditor, and without the transfer of money for interest.

If the several sums bear interest all could be paid at any time by discharging principal and accrued interest, and no one would lose.

The principle upon which the operations are based is very simple: If money is paid before it is due interest should be allowed upon it: if not paid until after it is due it should bear interest from maturity until date of payment.

PROBLEM.

"I owe $500, due in four months; $600, due in seven months, and $1000 due in nine months; what is the average term of credit?"

These sums do not bear interest until after maturity. If I should pay any one of them in full before its maturity I should lose the use of it for the given time. If I should not pay at maturity the creditor would be the loser.

I first experiment a little. Suppose I should pay them in full to-day; assuming money to be worth six per cent., what should I lose? On the first I should lose the interest for four months, or $10; on the second $21, and on the third $45. This is obviously true, for if I had the money in hand I could lend the different amounts for the several periods and realize the $76 as interest before the maturity of the debts. If I pay the debts to-day I shall pay out $2100. If I retain this sum until it earns me $76 at six per cent. and then pay it to my creditor, neither will lose. The interest on $2100 for one day at 6 per cent. is 35 cents. I should retain it as many days as $67 is times 35 cents.

It is not very material what day I select with which to make the "experiment." Select the most convenient date visible by simple inspection.

A problem in Average of Accounts is simply a double problem in Equation of Payments. First see that the pupils understand the meaning of the pro-

blem. It may be a transcript of a ledger account, and, consequently, not understood. The left-hand side shows amounts due the owner of the account from one whose name stands at the head of it, while the right-hand side shows the amounts due the one whose name is at the head, from the owner of the account.

Beginning with the debtor side, select some date and proceed as in the problem given so far as to ascertain the loss or gain if all be paid on the date selected Taking the same date perform the same experiment with the credit side.

Suppose the sum of the amounts on the debtor side to exceed the sum on the credit side by $100. This balance, then, is due the owner of the account. Suppose the interest items on the credit side, however, to exceed the similar items on the debtor side by $10. The meaning should be obvious. If the balance of the account be paid on the "trial date," the owner of the account will lose $10 of interest. Is the trial date too early or too late? How much, if the money is worth 6 per cent.?

CHAPTER XXIV.

EVOLUTION.

The only point under this head that will be discussed is the formation of the trial divisor in the extraction of cube root.

The rule as ordinarily stated is as follows:

"Take three times the square of the root already found, as a trial divisor."

By this method the work becomes very tedious if there are several terms in the root. Instead of the above, use the following:

Below the completed divisor write the square of the last term of the root. Find the sum of this square, the completed divisor, and the two additions made to the preceding trial divisor.

The following will illustrate the plan:

$\sqrt{869,457,256} = ?$

809,457,256 95

729

24300	140572
1350	
25	
25675	128375
25	12197256
27075	

By the old method the trial divisor would be found by squaring 95 and multiplying it by 3, by the present, all that is necessary is to write 25 below 25675, and find the sum of the 25, the 25675, the 25 and the 1350.

Pupils familiar with algebraic formular will readily understand the rule by examining the following:

$(a+b+c)^3=a^3+(3a^2+3ab+b^2)b+[3(a+b)^2+3(a+b)c+c^2]c$.

The first completed divisor is $3a^2+3ab+b^2$. The next trial divisor is $3(a+b)^2$, or $3a^2+6ab+3b^2$. This quantity exceeds the former by $3ab+2b^2$.

In the given problem $25675 = 3a^2 + 3a^1b + b^2$. $b^2 = 25$. $3ab = 1350$.

If, then, to 25675 we add the 25 below it, the 25 above it, and the 1350, we have increased it by $2b^2 + 3ab$.

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